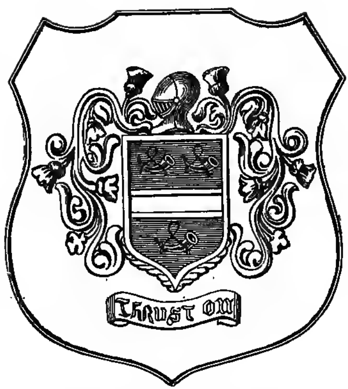
THRUST ON

ELEMENTARY TREATISE

ON

ANALYTICAL MECHANICS.

BY

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I. MANUAL OF ALGEBRA.

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IX. ANALYTICAL MECHANICS.

PREFACE.

THE following treatise has been prepared for use as a text-book in the School of Mines, but it is hoped that it may find a place in other Colleges and Schools of Science. It is intended to embrace all the principles of Analytical Mechanics that are needed by the student of Engineering, Architecture, and Geodesy. Its order of arrangement is the result of much practical experience, and its methods of demonstration have been thoroughly tested in the class-room.

In its development the methods of the Differential and Integral Calculus have been freely used, but not to the exclusion of the more elementary processes of Analysis. The plan adopted has been found to be well suited to Experimental Illustration, and the numerous practical examples scattered through the book have proved to be of great utility in impressing principles on the mind of the student.

For convenience of reference, the first equation on each page has been indicated by a number placed at the top of that page.

The fundamental principles of the Science are, for the most part, based on the axiomatic laws of Newton, but the more modern ideas of Work and Energy have been carefully considered and incorporated in the text. Goupillière's elegant method of treating the resistance of friction has been fully illustrated in the processes of finding the moduli of the elements of Mechanism.

It is believed that the order of arrangement will be found to be logical, the definitions clear and precise, and the demonstrations simple and comprehensive.

Columbia College, *July* 4, 1887.

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Note.—Names of Greek letters used in this book:

α. Alpha.	δ. Delta.	ρ. Rho.	ϕ. Phi.
β. Beta.	θ. Theta.	Σ. Sigma.	ω. Omega.
γ. Gamma.	π. Pi.	τ. Tau.	

MECHANICS.

I.—DEFINITIONS AND INTRODUCTORY PRINCIPLES.

Definition of a Body.

1. A **body** is a collection of material particles.

A body whose dimensions are exceedingly small is called a **material point.** In what follows the term *point* will generally be used in this sense.

Rest and Motion.

2. A point is **at rest** when it remains in the same relative position with respect to certain surrounding bodies that we regard as fixed; it is **in motion** when it continually changes this relative position.

The terms rest and motion as used in Mechanics are purely relative; it is probable that no point in the physical universe is absolutely at rest.

Force.

3. A **force** is that which tends to change the state of a body with respect to rest or motion.

If a body is at rest, whatever tends to set it in motion is a force; if in motion, whatever tends to make it move faster or

slower, or whatever tends to change the direction of its motion, is a force.

A force may act on a body for an instant and then cease, in which case it is called an **impulse**, or an **impulsive force**; or, it may act continuously, in which case it is called an **incessant force**.

For the purposes of mathematical investigation, an incessant force may be regarded as made up of a succession of impulses acting at equal but exceedingly small intervals of time. This interval, in the language of the calculus, may be denoted by dt, t being an independent variable and dt its constant differential. If the successive or elementary impulses are all equal the force is said to be **constant**, otherwise it is **variable**.

If forces act on a fixed body they produce *stress*, or *pressure;* if they act on a body that is free to move they produce *motion:* in the former case they are called **forces of pressure**, in the latter case they are called **moving forces**.

The same force under different circumstances may produce either pressure or motion: thus, if gravity act on a body that is supported it produces pressure, but if it acts on a body that is not supported it produces motion.

Definition of Mechanics.

4. Mechanics is the science that treats of the action of forces on bodies.

It is divided into two branches: **Statics**, which treats of the laws of pressure; and **Kinetics** (or *Dynamics*), which treats of the laws of motion.

In kinetics it is not motion alone that is considered, but the relation of forces to motion. That branch which treats of pure motion, without reference to the bodies moved or to the forces which produce the motion, is sometimes called **Kinematics**.

Equilibrium.

5. If the forces acting on a body balance each other, that is, if they counteract each other's effects, they are said to be **in equilibrium.**

Such a set of forces does not change the state of the body with respect to rest or motion; if the body is at rest it will remain so, or if in motion its motion will remain unchanged, so far as these forces are concerned.

When forces balance each other through the medium of a body at rest they are said to be in **statical equilibrium**; when they balance each other through the medium of a moving body they are in **dynamical equilibrium.**

Gravity, Weight, Mass, and Density.

6. The earth exercises an attractive force on bodies tending to draw them towards its centre. This force, which acts on every particle of a body, is called the **force of gravity.** If a body is supported, the force produces a pressure that is called the **weight** of the body. It is obvious then that the weight of a body varies as the quantity of matter it contains and as the force of gravity *conjointly.*

The **mass** of a body is the quantity of matter it contains. If we denote the weight of a body by W, its mass by M, and the force of gravity by g, we have, from what precedes,

$$W = Mg \quad \text{and} \quad M = \frac{W}{g} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The **density** of a body, or the degree of compactness of its particles, is proportional to the quantity of matter in a given volume. We may take as the measure of a body's density, its mass divided by its volume; or, denoting the density by d and the volume by V, we have

$$d = \frac{M}{V} \quad \text{and} \quad M = Vd \; . \; . \; . \; . \; . \; (2)$$

Combining (1) and (2), we have

$$W = Vdg \; . \; . \; . \; . \; . \; (3)$$

The **unit of weight** that we have adopted is the *avoirdupois pound* as determined by counterpoising it against the government standard, and the **unit of mass** is the quantity of matter in such a pound.

The force of gravity varies from point to point, but the weight of a body, as determined by a spring balance, varies in the same ratio; hence, the measure of its mass, $W \div g$, remains constant.

The **unit of density** is the density of distilled water at 39° F.

Uniform and Varied Motion.

7. The **velocity** of a moving point is its *rate* of motion. If a point moves over equal spaces in equal times its velocity is **constant** and its motion is **uniform**; otherwise its velocity is **variable** and its motion is **varied**. If the velocity continually *increases* the motion is **accelerated**; if the velocity continually *decreases* the motion is **retarded.**

The **unit of velocity** is assumed to be the velocity of a point which moves over *one foot* in *one second.* This is equivalent to the assumption that the *unit of length* is *one foot* and the *unit of time* is *one second.*

In uniform motion, the velocity of a body is the number of feet passed over by the body in one second; in varied motion the velocity at any instant is the number of feet that it would pass over in one second if its velocity were to remain unchanged for that time.

In uniform motion, if we denote the velocity by v, and the space passed over in t seconds by s, we have

$$v = \frac{s}{t} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

From equation (4), we have

$$s = vt, \quad \text{and} \quad t = \frac{s}{v} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Momentum.

8. The **momentum** of a body is its *quantity of motion.* It is obvious that the quantity of motion of a body varies conjointly with the mass of the body and with its velocity; hence, the measure of the body's momentum is equal to mv.

The **unit of momentum** is the momentum of a unit of mass, moving with a unit of velocity.

Work, Energy.

9. A force is said to perform **work** when it overcomes a resistance. Any kind of work may be assimilated to that of raising a weight, as in lifting a bucket of water from a well. In this case it is obvious that the work performed varies conjointly as the weight raised and as the height through which it is raised. Denoting the weight in pounds by W and the height in feet by h, we have for the measure of the work performed Wh.

The **unit of work** is the work required to raise a weight of *one pound* through a height of one *foot.* This unit, which is termed a *physical unit,* is denoted by the symbol 1 *ft. lb.*, and is called a **foot pound.** Thus, the work required to raise 7 *lbs.* through 5 *ft.* is equal to 35 *ft. lbs.*, that is, to 35 *foot pounds.*

The **rate of work** of a force is the number of units of work it can perform in a given time. In Mechanism, the *rate* is usually expressed in terms of a **horse power,** which is technically assumed to be 33,000 *ft. lbs.* per minute or 550 *ft. lbs.* per second. Thus, an engine of 10 horse power is

one that is capable of performing 5,500 units of work per second.

Energy is the capacity of a body to perform work : it may be **potential**, or **kinetic**. Thus, the weight of a clock when wound up has a certain amount of *potential energy* which is utilized, as the weight runs down, to keep the clock train in motion ; a moving body, as the falling hammer of a pile-driver, has an amount of *kinetic energy* which is utilized in performing the work of driving the pile into the ground.

Geometrical Representation of a Force.

10. A force is said to be *given* when we know its **intensity** or *magnitude ;* its **point of application**, that is, the point at which it is supposed to act ; and its **line of action**, that is, the direction in which it tends to move the point of application.

A force is represented geometrically by a straight line whose length is proportional to its intensity ; one extremity as O, represents the point of application and an arrow head shows the direction in which the force is supposed to act.

Fig. 1.

If a force is applied to a **solid body**, that is, to a body whose particles are *rigidly connected*, the point of application may be taken at any point on the line of action of the force.

Measure of Forces.

11. A force is measured by comparing it with another force of the same kind taken as a *unit*.

Pressures are measured in pounds, the **unit** being *one avoirdupois pound*. When we speak of a pressure of n pounds we mean a force which, if directed vertically upward, would just sustain a weight of n standard pounds.

Moving forces are measured in terms of the *momenta they can generate*.

The **unit of an impulsive force** is an impulse that can **generate a unit of momentum.**

The **unit of a constant force** is a constant force which acting on a unit of mass for a unit of time can **generate a unit of momentum.**

The measure of a variable force at any instant is the momentum it could generate in a unit of time if it were to remain constant for that time.

The unit of pressure may be called a *statical* unit, and the unit of an incessant moving force may be called a *kinetic unit.* Both of these units are of the *same kind:* for, from Art. 6 the statical unit, *one pound,* is equal to a *unit of mass* multiplied by g; but it will be shown hereafter that the force of gravity acting on a unit of mass for a unit of time will generate about 32.16 units of momentum: hence the statical unit is about 32.16 times the kinetic unit. All kinds of forces may therefore be expressed in terms the kinetic unit.

The Newtonian Laws of Motion.

12. The following three laws, commonly known as the **Newtonian Laws,** which are deduced from universal experience, are accepted as *axiomatic* in treating of the motions of bodies:

FIRST LAW. *Every body continues in a state of rest or of uniform motion in a straight line until compelled by impressed forces to change that state.*

SECOND LAW. *Change of motion is proportional to the impressed force, and takes place in the direction of the straight line in which the force acts.*

THIRD LAW. *To every action there is always an equal and contrary reaction: or, the mutual actions of any two bodies are always equal and oppositely directed.*

The *first law* is equivalent to the assertion that no body has power of itself to change its state with respect to rest or motion.

In the *second law* the term *change of motion* is equivalent

to the expression *change of momentum*. This law is equivalent to the assertion that of several forces acting on a body each produces its own effect as though the others did not exist, and this whether the body is at rest or in motion. If equal and opposite forces act on a body the law still holds good: thus, if a boat is moving up a river with a given velocity and a body is compelled to move from stem to stern with an equal velocity, we may regard the body as acted upon by equal and opposite forces, each of which produces its own effect; but these effects balance each other so that the body remains at rest with respect to objects on shore.

The *third law* holds true whether the forces considered are statical or kinetic. If a heavy body exerts a downward pressure on a table, the table exerts an equal upward pressure; if the earth exercises an attraction on a stone drawing it downward, the stone exerts an attraction on the earth drawing it upward, the momentum generated in each case being the same; if a moving body A impinges upon a body B which is free to move, A imparts some of its momentum to B, and the effect on A is the same as though it were acted upon by a force equal and opposite to that exerted by A upon B.

II.—COMPOSITION AND RESOLUTION OF FORCES.

Definitions.

13. Composition of forces, is the operation of finding a single force whose effect is the same as that of two or more given forces. The required force is called the **resultant** of the given forces.

Resolution of forces, is the operation of finding two or more forces whose combined effect is equivalent to that of a given force. The required forces are called **components** of the given force.

Composition of Forces whose directions coincide.

14. From the rules laid down for measuring forces, it follows, that the resultant of two forces applied at a point, and acting in the same direction, is equal to the sum of the forces. If two forces act in opposite directions, their resultant is equal to their difference, and it acts in the direction of the greater.

If any number of forces be applied at a point, some in one direction and others in a contrary direction, their resultant is equal to the sum of those that act in one direction, diminished by the sum of those that act in the opposite direction, or calling those that act in one direction *plus* and those that act in a contrary direction *minus*, the resultant is equal to the algebraic sum of the components.

Forces that have a common line of action are called **homologous**; their algebraic sum may be indicated by writing the expression for one of the forces in a parenthesis and prefixing

the symbol Σ. Thus, if we denote the resultant of the group by R and one of the homologous components by P, we have

$$R = \Sigma (P), \quad . \quad . \quad . \quad . \quad . \quad (6)$$

which is read, *The resultant of a set of homologous forces is equal to their algebraic sum.*

The forces treated of in this article and also in the following articles, are supposed to be applied at points of a solid body. When their lines of action intersect, they are said to be **concurrent.**

Parallelogram of Forces.

15. Suppose two forces, P and Q, to be applied to a solid body at a point O and let them be represented in direction and intensity by OP and OQ: complete the parallelogram PQ and draw its diagonal OR.

Let the body contain a unit of mass; then if the forces are impulses OP and OQ will represent the velocities generated by P and Q (Art. 11), and inasmuch as each produces its own effect as though the other did not exist (Art. 12), the body will be found at the end of one second somewhere on PR by virtue of the force P and somewhere on QR by virtue of the force Q; it will therefore be at R. Had O been acted on by an impulse represented by OR, it would in like manner have moved from O to R in one second. Hence the impulse OR is equivalent in effect to the two impulses OP and OQ; that is,

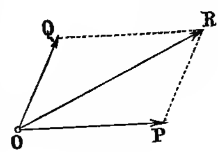

Fig. 2.

If two impulsive forces be represented by adjacent sides of a parallelogram, their resultant will be represented by that diagonal of the parallelogram which passes through their common point.

If the forces are constant forces, OP and OQ will represent

the velocities they can generate in a unit of time, and for the same reason as before OR will represent a constant force equivalent in effect to the forces P and Q: hence, the principle holds true for constant forces. It is also true for forces of pressure; for, if we apply a force equal and directly opposed to the resultant of the two moving forces it will hold them in equilibrium, converting them into forces of pressure, but it will in no manner change the relation between them and their resultant. Hence, the principle holds for all kinds of forces: it may be enunciated as follows:

If two forces be represented in direction and intensity by adjacent sides of a parallelogram, their resultant will be represented by that diagonal of the parallelogram which passes through their common point.

This principle is called **the parallelogram of forces.**

Geometrical Applications of the Parallelogram of Forces.

16. 1°. Given two forces; to find their resultant.

Let OP and OQ be the given forces. Complete the parallelogram QP and draw its diagonal OR; this will be the resultant required.

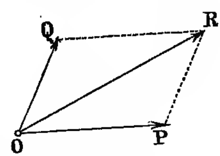

Fig. 3.

2°. Given, a force and one of its components; to find the other.

Let OR be a force and OP one of its components. Draw PR and complete the parallelogram PQ; OQ will be the other component.

3°. Given, a force and the directions of its components; to find the components.

Let OR be a force and OP, OQ, the directions of its components; through R draw RQ and RP parallel to PO and QO; then will OP and OQ be the required components.

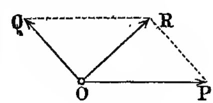

Fig. 4.

4°. Given, a force and the intensities of its components; to find the directions of the components.

Let OR be a force, and let the intensities of its components be represented by lines equal to OP and OQ; with O as a centre and OP as a radius, describe an arc, then with R as a centre and OQ as a radius, describe a second arc, cutting the first at P; draw OP, and RP, and complete the parallelogram PQ; OP and OQ will be the required components.

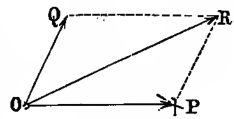

Fig. 5.

Polygon of Forces.

17. Let OQ, OP, OS, and OT, be a system of forces applied at a point, O, and lying in a single plane. To construct their resultant; on OQ and OP construct the parallelogram PQ, and draw its diagonal OR', this will be the resultant of OP and OQ. In like manner construct a parallelogram on OR' and OS; its diagonal OR'', will be the resultant of OP, OQ, and OS. On OR'' and OT construct a parallelogram, and draw its diagonal OR; then will OR be the resultant of all the given forces. This method of construction may be extended to any number of forces whatever.

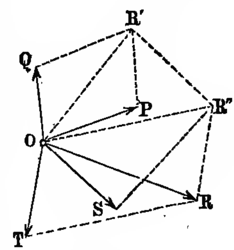

Fig. 6.

If we examine the diagram, we see that QR' is parallel and equal to OP, $R'R''$ is parallel and equal to OS, $R''R$ is parallel and equal to OT, and that OR is drawn from the point of application, O, to the extremity of $R''R$. Hence, we have the following rule for constructing the resultant of several concurrent forces:

Through their common point draw a line parallel

and equal to the first force; through the extremity of this draw a line parallel and equal to the second force; and so on, throughout the system; finally, draw a line from the starting point to the extremity of the last line drawn, and this will be the resultant required.

This application of the parallelogram of forces is called **the polygon of forces.**

The construction holds true, even when the forces are not in one plane. In this case, the lines OQ, QR', $R'R''$, &c., form a **twisted polygon,** that is, a polygon whose sides are not in one plane.

When the point R, in the construction, falls at O, OR reduces to 0, and the forces are in equilibrium.

The simplest case of the preceding principle is the **triangle of forces.** Thus, to find the resultant of P and Q (Fig. 3), we draw OP parallel and equal to the first force, and from P we draw PR parallel and equal to the second force; then the line OR is the required resultant. If three forces are parallel and equal to the three sides of a triangle *taken in order*, the forces are in equilibrium.

Parallelopipedon of Forces.

18. Let OP, OQ, and OS, be three concurrent forces not in the same plane. On these, as edges, construct the parallelopipedon OR, and draw OR, OM, and SR. From the principle of Art. 15, OM is the resultant of OP and OQ; and OR is the resultant of OM and OS; hence, OR is the resultant of OP, OQ, and OS; that is,

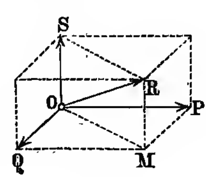

Fig. 7.

If three forces be represented by the concurrent edges of a parallelopipedon, their resultant will be repre-

sented by the diagonal of the parallelopipedon that passes through their common point.

This principle is called the **parallelopipedon of forces.**

It is easily shown that it is a particular case of *the polygon of forces;* for, OP is parallel and equal to the first, PM to the second, MR to the third force, and OR is drawn from the origin, O, to the extremity of MR.

Components of a Force in the direction of Rectangular Axes.

19. *First.* To find analytical expressions for the components of a force in the direction of two axes.

Let AR be a force in the plane of the rectangular axes OX and OY. On it as a diagonal construct a parallelogram ML, whose sides are parallelogram to OX and OY. Denote AR by R, AL by X, AM, equal to LR, by Y, and the angle LAR, equal to the angle the force makes with OX, by α. From the figure, we have

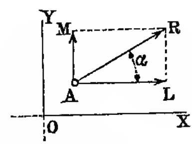

Fig. 8.

$$X = R \cos \alpha, \text{ and } Y = R \sin \alpha \quad . \quad . \quad . \quad . \quad . \quad (7)$$

In these expressions the angle α is estimated from the positive direction of the axis of X, around to the force, in accordance with the rule laid down in Trigonometry. The component X will have the same sign as $\cos \alpha$, and the component Y the same sign as $\sin \alpha$.

Secondly. To find the components of a force in the direction of three rectangular axes.

Let OR, denoted by R, be the given force, and OX, OY, and OZ, the given axes. On OR, as a diagonal, construct a parallelopipedon whose edges are parallel to the axes.

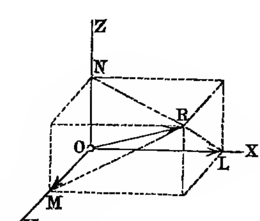

Fig. 9.

Then will OL, OM, and ON be the required components. Denote these by X, Y, and Z, and the angles they make with OR by α, β, and γ. Join R with L, M, and N, by straight lines. From the right-angled triangles thus formed, we have

$$X = R\cos\alpha,\ Y = R\cos\beta, \text{ and } Z = R\cos\gamma \quad . \quad . \quad . \quad (8)$$

The angles α, β, and γ are estimated from the positive directions of the corresponding axes, as in Trigonometry, and each component has the same sign as the corresponding cosine.

If a force be resolved in the direction of rectangular axes, each component will represent the total effect of the given force in that direction. For this reason such components are called **effective components.** It is plain that the component in the direction of each axis is the same as the **projection** of the force on that axis, the projection being made by lines through the extremities of the force, and perpendicular to the axis. Hence, we may find the effective component of a force in the direction of a given line, by multiplying the force into the cosine of its inclination of the line.

It is obvious that the projection of the resultant of two or more forces on a given line is equal to the algebraic sum of the projections of its components on that line. It is also obvious that the resultant of two forces is equal to the sum of the projections of the forces on the direction of the resultant.

Analytical Composition of Rectangular Forces.

20. *First.* When there are but two forces.

Let AL and AM be rectangular forces, denoted by X and Y, and let AR, denoted by R, be their resultant. Denote the angle LAR by α. Then, because $LR = Y$, we have, from the triangle ALR,

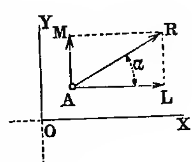

Fig. 10.

$$R = \sqrt{X^2 + Y^2}; \ \cos\alpha = \frac{X}{R}; \text{ and } \sin\alpha = \frac{Y}{R} \quad . \quad . \quad . \quad . \quad (9)$$

The first of these gives the intensity, the second and third the direction of the resultant.

Secondly. When there are three forces not in one plane. Let OL, OM, and ON, be rectangular forces denoted by X, Y, and Z, and let OR, denoted by R, be their resultant. Denote the angles which R makes with OL, OM, and ON by α, β, and γ. Then, from the figure, we have

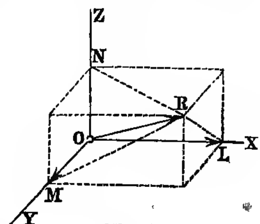

Fig. 11.

$$R = \sqrt{X^2 + Y^2 + Z^2} \quad . \quad . \quad . \quad (10)$$

$$\cos\alpha = \frac{X}{R}; \ \cos\beta = \frac{Y}{R}; \text{ and, } \cos\gamma = \frac{Z}{R} \quad . \quad . \quad . \quad . \quad . \quad (11)$$

The first gives the intensity of the resultant, the others its direction.

Examples.

1. Two pressures of 9 and 12 pounds act on a point, and at right angles to each other. Required, the resultant pressure.

Ans. The resultant pressure is 15 *lbs.*, and it makes an angle of 53° 7′ 47″ with the direction of the first force.

2. Two rectangular forces are to each other as 3 to 4, and their resultant is 20 *lbs.* What are the intensities of the components?

Ans. $X = 12$ *lbs.*, and $Y = 16$ *lbs.*

3. A boat fastened by a rope to a point on shore, is urged by the wind perpendicular to the current, with a force of 18 pounds, and down the current by a force of 22 pounds. What is the tension on the rope, and what angle does it make with the current?

Ans. $R = 28.425$ *lbs.*; $a = 39° \ 17′ \ 20″$.

4. Required the intensity and direction of the resultant of three forces at right angles to each other, having the intensities 4, 5, and 6 pounds, respectively.

Ans. $R = \sqrt{16 + 25 + 36} = 8.775$ *lbs.*; and $a = 62° 52' 51''$; $\beta = 55° 15' 50''$; $\gamma = 46° 51' 43''$.

5. Three forces at right angles are to each other as 2, 3, and 4, and their resultant is 60 *lbs.* What are the intensities of the forces?

Ans. 22.284 *lbs.*, 33.426 *lbs.*, and 44.568 *lbs.*

Application to Groups of Concurrent Forces.

21. The principles explained in the preceding articles, enable us to find the resultant of any number of concurrent forces. Let P, P', P'', &c., be a group of concurrent forces. Call the angles they make with the axis of X, α, α', α'', &c.; the angles they make with the axis of Y, β, β', β'', &c.; and the angles they make with the axis of Z, γ, γ', γ'', &c. Resolve each force into rectangular components parallel to the axes, and denote the resultants of the groups parallel to the axes by X, Y, and Z. We have, (Art. 19),

$$X = \Sigma\,(P\cos\alpha),\ \ Y = \Sigma\,(P\cos\beta),\ \ Z = \Sigma\,(P\cos\gamma).$$

If we denote the resultant by R, and the angles it makes with the axes by a, b, and c, we have, as in Article 20,

$$R = \sqrt{X^2 + Y^2 + Z^2}.$$

$$\cos a = \frac{X}{R},\ \cos b = \frac{Y}{R},\ \text{and}\ \cos c = \frac{Z}{R}\ \ .\ .\ .\ .\ .\ \ (12)$$

When the given forces lie in the plane XY, Z reduces to 0, $\cos\beta$ becomes $\sin\alpha$, $\cos b$ becomes $\sin a$, and the formulas reduce to

$$X = \Sigma\,(P\cos\alpha),\ \text{and}\ Y = \Sigma\,(P\sin\alpha).$$

$$R = \sqrt{X^2 + Y^2},\ \text{and}\ \cos a = \frac{X}{R},\ \text{and}\ \sin a = \frac{Y}{R}\ \ .\ .\ .\ \ (13)$$

Examples.

1. Three concurrent forces, whose intensities are 50, 40, and 70, lie in the same plane, and make with an axis, angles equal to 15°, 30°, and 45°. Required the resultant.

Here, $X = 50 \cos 15° + 40 \cos 30° + 70 \cos 45° = 132.435$,

and $Y = 50 \sin 15° + 40 \sin 30° + 70 \sin 45° = 82.44$.

Ans. $R = 156$, and $A = 31° \; 54' \; 12''$.

2. Three forces, 4, 5, and 6, lie in the same plane, and make equal angles with each other. Required the intensity of their resultant and the angle it makes with the least force.

Ans. $R = \sqrt{3}$, and $a = 210°$.

3. Two forces, one of 5 *lbs.* and the other of 7 *lbs.*, are applied at the same point, and make with each other an angle of 120°. What is the intensity of their resultant. *Ans.* 6.24 *lbs.*

Formula for the Resultant of Two Forces.

22. Let P and P', be two concurrent forces, and let the axis of X be taken to coincide with P; α will then be 0, and we shall have $\sin \alpha = 0$, and $\cos \alpha = 1$. The value of X will be $P + P' \cos \alpha'$, and the value of Y will be $P' \sin \alpha'$. Squaring these values, substituting in Equation (9), and reducing by the relation $\sin^2 \alpha' + \cos^2 \alpha' = 1$, we have

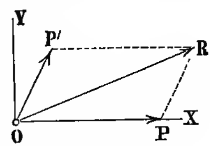

Fig. 12.

$$R = \sqrt{P^2 + P'^2 + 2PP' \cos \alpha'} \quad . \; . \; . \; . \; . \quad (14)$$

The angle α' is the angle between the given forces. Hence,

The resultant of two concurrent forces is equal to the square root of the sum of the squares of the forces, plus twice the product of the forces into the cosine of their included angle.

If $\alpha' = 0$, we have $R = P + P'$.

If $\alpha' = 180°$, we have $R = P - P'$.

EXAMPLES.

1. Two forces, P and Q, are equal to 24 and 30, and the angle between them is 105°. What is the intensity of their resultant ?

Ans. $R = 32.21$.

2. Two forces, P and Q, whose intensities are 5 and 12, have a resultant whose intensity is 13. Required the angle between them.

Ans. Cos $a = 0$, or $a = 90°$.

3. A boat is impelled by the current at the rate of 4 miles per hour, and by the wind at the rate of 7 miles per hour. What will be her rate per hour when the direction of the wind makes an angle of 45° with that of the current? *Ans.* $R = 10.2$m.

4. Two forces, and their resultant, are all equal. What is the angle between the two forces? *Ans.* 120°.

5. Two forces at right angles are to each other as 1 is to $\sqrt{3}$, and their resultant is 10 *lbs.* What are the forces? *Ans.* 5 and $5\sqrt{3}$.

6. If three forces are to each other as 1, 2, and 3, and are in equilibrium, show that the forces act along the same straight line.

Relation Between Two Forces and their Resultant.

23. Let P and Q be two forces, and R their resultant. Then because QP is a parallelogram, the side PR is equal to Q. From the triangle ORP, we have,

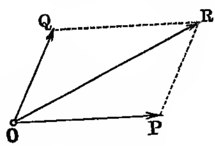

Fig. 13.

$$P : Q : R :: \sin ORP : \sin ROP : \sin OPR.$$

But,

$$ORP = QOR, \text{ and } OPR = 180° - QOP;$$

hence,

$$P : Q : R :: \sin QOR : \sin ROP : \sin QOP; \quad . \quad . \quad . \quad . \quad (15)$$

That is, *of two forces and their resultant, each is proportional to the sine of the angle between the other two.*

If we apply a force R' equal and directly opposed to R, the forces P, Q, and R' will be in equilibrium. The angle QOR is the supplement of QOR', and POR is the supplement of POR'; hence,

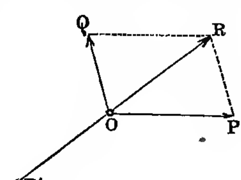

Fig. 14.

$$\sin QOR = \sin QOR',$$

and $$\sin POR = \sin POR';$$

we have also, $R = R'$.

Making these substitutions in the preceding proportion, we have,

$$P : Q : R' :: \sin QOR' : \sin POR' : \sin QOP \quad . \quad . \quad . \quad . \quad (16)$$

Hence, *if three forces are in equilibrium, each is proportional to the sine of the angle between the other two.*

Examples.

1. A weight of 50 *lbs.*, suspended by a string, is drawn aside by a horizontal force until the string makes an angle of 30° with the vertical. Required the value of the horizontal force, and the tension of the string.
Ans. 28.8675 *lbs.*, and 57.735 *lbs.*

2. A point is kept in equilibrium by three forces of 6 *lbs.*, 8 *lbs.*, and 11 *lbs.*; what are the angles between the forces?
Ans. 77° 21′ 52″, 147° 50′ 34″, and 134° 47′ 34″.

3. A weight of 25 *lbs.* is attached to the ends of two strings whose lengths are 3 and 4 *ft.*, the other ends of the strings being attached at points of a horizontal beam which are 5 *ft.* apart. What are the tensions of the strings? *Ans.* 20 *lbs.*, and 15 *lbs.*

Translation; Rotation; Principle of Moments.

24. A body is said to have a **motion of translation** when all of its points move in parallel straight lines: it has a **motion of rotation** when its points move in arcs of circles whose centres are in a straight line; this line is called the **axis of rotation**, and any plane perpendicular to it is called a **plane of rotation.**

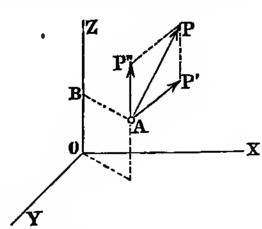

Fig. 15.

If a body is restrained by a fixed axis, any force applied to it, whose line of action does not intersect the axis, will tend to impart a motion of rotation.

Let P be a force acting on a body which is restrained by an axis OZ. Draw a line AB, perpendicular to the force and also to the axis. Let A be taken as the point

of application of the force, and at this point resolve it into two components P'' and P', the former *parallel*, and the latter *perpendicular* to OZ. The component P'' can have no tendency to produce rotation about the axis; hence, P', which is the projection of P on a plane of rotation, is the only component that is effective in producing rotation.

It is obvious that the tendency of P' to produce rotation varies conjointly with P' and with AB; we may therefore take the product $P' \times AB$ as the measure of the tendency of the force P to produce rotation around OZ: this product is called a **moment.** Hence, the moment of a force with respect to an axis is the product of the projection of the force on a plane of rotation by the distance of the force from the axis. Let us first consider the case in which the forces lie in a plane of rotation.

Let P and Q be two forces applied at O, and let R be their resultant. Suppose also that O is a point of a solid body which is restrained by an axis perpendicular to the plane POQ at C; from C draw perpendiculars to P, Q, and R, denoting them respectively by p, q, and r.

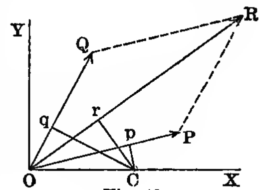

Fig. 16.

Take the line OC as the axis of X, and the perpendicular OY as the axis of Y; denote the angle XOP by α, XOQ by β, and XOR by ϕ. Because R is the resultant of P and Q, the projection of R on the axis of Y is equal to the sum of the projections of P and Q on the same axis (Art. 19), that is,

$$R \sin \phi = P \sin \alpha + Q \sin \beta,$$

or, multiplying through by OC, we have,

$$R\,(OC \sin \phi) = P\,(OC \sin \alpha) + Q\,(OC \sin \beta);$$

but, $OC \sin \phi = r$, $OC \sin \alpha = p$, and $OC \sin \beta = q$;

whence, $$Rr = Pp + Qq \quad . \quad . \quad . \quad . \quad (a)$$

The signs of r, p, and q, depend upon the values of ϕ, α, and β. If, for example, C falls within the angle POR, sin α is *negative*, and equation (a) becomes

$$Rr = Qq - Pp.$$

If C falls on R, $\sin \phi = 0$, sin α is negative, and we have,

$$0 = Qq - Pp, \quad \text{or} \quad Pp = Qq \quad . \quad . \quad . \quad . \quad . \quad (b)$$

Hence, in all cases, *the moment of the resultant of two forces is equal to the algebraic sum of the moments of the forces taken separately.*

If we regard the force Q as the resultant of two others, and one of these in turn, as the resultant of two others, and so on, the principle may be extended to any number of concurrent forces in the same plane. This principle may be expressed by the equation,

$$Rr = \Sigma (Pp) \quad . \quad . \quad . \quad . \quad . \quad (17)$$

That is, *the moment of the resultant of any number of concurrent forces, in the same plane, is equal to the algebraic sum of the moments of the forces taken separately.*

This is **the principle of moments.**

The moment of the resultant is the **resultant moment**; the moments of the components are **component moments**; and the plane passing through the resultant and centre of moments, is the **plane of moments.**

It is plain, from what precedes, that the moment of the resultant of a group of concurrent forces in space with respect to an axis may be found by projecting the forces on a plane of rotation and taking the algebraic sum of the moments of the projections with respect to the point in which this plane cuts the axis as a centre of moments.

We have seen that the moment of a force may be either

positive or negative. If rotation in one direction is taken as *positive*, rotation in the opposite direction must be regarded as *negative*. In what follows we adopt the following conventional rule: If, to an eye in the positive direction of the axis and without the body, the force tends to turn the body in the same direction as the motion of the hands of a clock, we call its moment *positive*; if in a contrary direction, *negative*.

In the diagram the moments of P, Q, and R, are all positive. If C is taken on R, the moment of P is negative, and the moment of Q is positive. These moments, as we see from equation (b), balance each other. In general, if the moment of the resultant is 0, the positive moments balance the negative ones, and there is no resultant rotation.

Resultant of Parallel Forces.

25. Let P and Q be two forces in the same plane, and applied at points invariably connected, as, for example, at the points M and N of a solid body, and let P be the greater. Their lines of direction, when prolonged, will meet at some point O; and if we suppose the points of application to be transferred to O, the resultant may be determined by the parallelogram of forces.

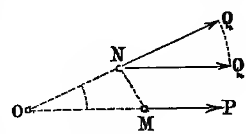

Fig. 17.

Let this resultant be denoted by R, and let the angles between it and its components P and Q be denoted by α and β. Then, from Art. 19, we shall have

$$R = P\cos\alpha + Q\cos\beta \quad . \quad . \quad . \quad . \quad . \quad (a)$$

1°. If O falls to the left of M the resultant will pass through O, and will in this case cut MN at some point between M and N.

2°. If O falls to the right of M the resultant will pass

through O, and will in this case cut the prolongation of MN on the side of M.

In the *first* case, suppose Q to turn around N, as indicated in Fig. 17; the point O will recede to the left, and when Q becomes parallel to P, the resultant will become parallel to P; α and β will become 0, and from equation (a) we shall have

$$R = P + Q.$$

That is, *if two forces are parallel and in the same direction, their resultant will be parallel to both, and in the same direction; it will lie between the two forces: and will be equal to their sum.*

In the *second* case, suppose Q to turn around N, as indicated in Fig. 18; the point O will then recede to the right, and when Q becomes parallel to P, the resultant will become parallel to P; α will become 0, and β will become 180°, and from equation (a) we shall have

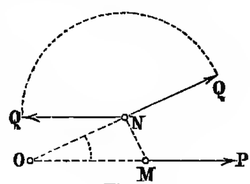

Fig. 18.

$$R = P - Q.$$

That is, *if two forces are parallel and in opposite directions, their resultant will be parallel to both, and will act in the direction of the greater; it will lie outside of both on the side of the greater; and will be equal to the difference of the two forces.*

To find the point at which R cuts MN; let P and Q be the parallel forces applied at M and N, and let S be the point in which R cuts MN. Through N draw NL perpendicular to the general direction of the forces, and assume the point C, in which it intersects the line of direction of R, as a centre

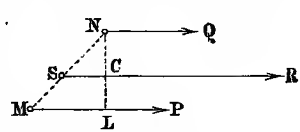

Fig. 19.

of moments. Since the centre of moments is on the line of direction of the resultant, the lever arm of the resultant will be 0, and we shall have, from the principle of moments (Art. 24),

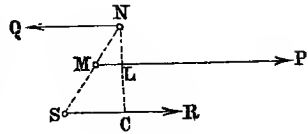

Fig. 20.

$$P \times CL = Q \times CN;$$

or, $$P : Q :: CN : CL.$$

But, from the similar triangles CNS and LNM, we have,

$$CN : CL :: SN : SM.$$

Combining the two proportions, we have,

$$P : Q :: SN : SM.$$

That is, *the line of direction of the resultant divides the line joining the points of application of the components, inversely as the components.*

From the last proportion, we have, by composition,

$$P : Q : P + Q :: SN : SM : SN + SM;$$

and, by division,

$$P : Q : P - Q :: SN : SM : SN - SM.$$

In the *first* case (Fig. 19), $P + Q = R$, and $SN + SM = MN$; in the *second* case (Fig. 20), $P - Q = R$, and $SN - SM = MN$. Substituting in the preceding proportions, we have,

$$P : Q : R :: SN : SM : MN;$$

or, $$P : Q : R :: CN : CL : NL \quad . \; . \; . \; . \; . \quad (18)$$

That is, *of two parallel forces and their resultant, each is proportional to the distance between the other two.*

Two *equal* parallel forces acting in contrary directions but

not directly opposed constitute a **couple**; the distance between them is the **lever arm** of the couple. From what precedes we see that the resultant of a couple is 0, and that its point of application is at an infinite distance. It is easily shown that the tendency of a couple to produce rotation around an axis perpendicular to the plane of the forces is equal to *either force into the lever arm of the couple.*

Composition of Parallel Forces.

26. 1°. To find the resultant of two parallel forces lying in the same direction:

Let P and Q be the forces, M and N their points of application. Make $MQ' = Q$, and $NP' = P$; draw $P'Q'$ cutting MN in S; through S draw SR parallel to MP, and make it equal to $P + Q$; it will be the resultant.

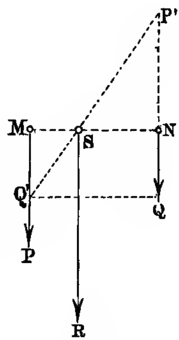

Fig. 21.

For, from the triangles $P'SN$ and $Q'SM$, we have,

$$P'N : Q'M :: SN : SM; \quad \text{or,} \quad P : Q :: SN : SM.$$

2°. To find the resultant of two parallel forces acting in opposite directions:

Let P and Q be the forces, M and N their points of application. Prolong QN till $NA = P$, and make $MB = Q$; draw AB, and produce it till it cuts NM produced in S; draw SR parallel to MP, and equal to BP, it will be the resultant required.

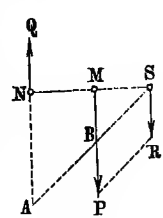

Fig. 22.

For, from the triangles SNA and SMB, we have,

$$AN : BM :: SN : SM; \quad \text{or,} \quad P : Q :: SN : SM.$$

These constructions, which are essentially the same, suggest the methods of resolving a force into two parallel components, applied at given points.

3°. To find the resultant of any number of parallel forces:

Let P, P', P'', P''', be parallel forces. Find the resultant of P and P', by the rule already given, it will be $R' = P + P'$; find the resultant of R' and P'', it will be $R'' = P + P' + P''$; find the resultant of R'' and P''', it will be $R = P + P' + P'' + P'''$. If there be a greater number of forces, the operation of composition may be continued; the final result will be the resultant of the system. If some of the forces act in contrary directions, combine all that act in one direction, as just explained, and call their resultant R'; then combine all that act in a contrary direction, and call their resultant R''; finally, combine R' and R''; their resultant, R, will be the resultant of the system.

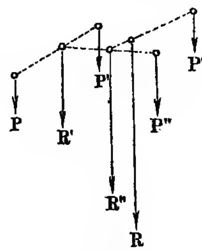

Fig. 23.

If the forces P, P', etc., be turned about their points of application, their intensities remaining unchanged, and their directions remaining parallel, the forces R', R'', R, will also turn about fixed points, continuing parallel to the given forces. The point through which R always passes is called the **centre of parallel forces.**

Co-ordinates of the Centre of Parallel Forces.

27. Let P, P', P'', etc., be parallel forces, applied at points that maintain fixed positions with respect to a system of rectangular axes, and let R, equal to $\Sigma(P)$, be their resultant. Denote the co-ordinates of the points of application of the forces by x, y, z; x', y', z', etc.; and those of R by x_1, y_1, z_1.

Turn the forces about their points of application, till they are parallel to the axis of Y, and in that position find their moments with respect to the axis of Z. In this position the lever arms of the forces are x, x', etc., and the lever arm of R is x_1. From the principle of moments (Art. 24), we have,

$$Rx_1 = Px + P'x' +, \text{ etc.}$$

or,

$$x_1 = \frac{\Sigma (Px)}{\Sigma (P)} \quad . \quad . \quad . \quad . \quad . \quad (19)$$

By making the forces in like manner parallel to the axis of Z, and taking their moments with respect to the axis of X, we have,

$$y_1 = \frac{\Sigma (Py)}{\Sigma (P)} \quad . \quad . \quad . \quad . \quad . \quad (20)$$

And in like manner, we find,

$$z_1 = \frac{\Sigma (Pz)}{\Sigma (P)} \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Forces Applied at Points Invariably Connected.—Tendency to Produce Translation.

28. Let P, P', P'', etc., be forces situated in any manner in space and applied at points that are invariably connected, as, for example, to points of the same solid body. Assume any point O, and through it draw any three lines, OX, OY, and OZ, at right angles to each other, and take these lines as axes. Denote the angles that P, P', P'', etc., make with the axis of X by α, α', α'', etc.; the angles they make with the axis of Y by β, β', β'', etc.; and the angles they make with the axis of Z by γ, γ', γ'', etc.; also denote the co-ordinates of their points of application by x, y, z;

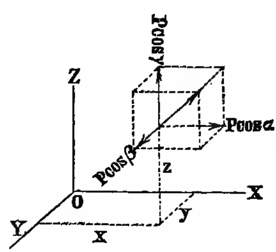

Fig. 24.

x', y', z'; x'', y'', z'', etc. Then let each force be resolved into components parallel to the axes, as shown in Fig. 24.

We have for the group parallel to the axis of X,

$$P \cos \alpha, \ P' \cos \alpha', \ P'' \cos \alpha'', \text{ etc.};$$

for the group parallel to the axis of Y,

$$P \cos \beta, \ P' \cos \beta', \ P'' \cos \beta'', \text{ etc.};$$

and, for the group parallel to the axis of Z,

$$P \cos \gamma, \ P' \cos \gamma', \ P'' \cos \gamma'', \text{ etc.}$$

Denoting the resultants of these several groups by X, Y, and Z, we have

$$X = \Sigma\,(P \cos \alpha), \ \ Y = \Sigma\,(P \cos \beta), \text{ and } Z = \Sigma\,(P \cos \gamma).$$

Now, when the system of forces does not reduce to a *couple*, it must have a single resultant, that is, X, Y, and Z must intersect at a common point whose co-ordinates may be denoted by x_1, y_1, and z_1, the values of which may be found as in the last article.

Forces Applied at Points Invariably Connected.—Tendency to Produce Rotation.

29. Continuing the supposition of the last article, we see that X acts with a lever arm y_1, to produce negative rotation around the axis of Z, and that Y acts with a lever arm x_1 to produce positive rotation around the same axis; hence, the entire tendency to rotation around the axis of Z is

$$Yx_1 - Xy_1.$$

But $Yx_1 = \Sigma\,(P \cos \beta\, x)$ and $Xy_1 = \Sigma\,(P \cos \alpha\, y)$; hence, the preceding expression becomes

$$\Sigma\,(P \cos \beta\, x) - \Sigma\,(P \cos \alpha\, y).$$

In like manner it may be shown that the tendency to pro-

duce rotation around the axis of X is measured by the expression

$$\Sigma\,(P\cos\gamma\,y) - \Sigma\,(P\cos\beta\,z).$$

Also, the tendency to rotation around the axis of Y is measured by the expression,

$$\Sigma\,(P\cos\alpha\,z) - \Sigma\,(P\cos\gamma\,x).$$

Equilibrium.

30. The system will be in equilibrium when there is no tendency to motion of translation in the direction of either axis, and when there is no tendency to rotation around either axis, hence we must have (Arts. 28, 29),

$$\left.\begin{aligned}\Sigma\,(P\cos\alpha) &= 0,\\ \Sigma\,(P\cos\beta) &= 0,\\ \Sigma\,(P\cos\gamma) &= 0.\end{aligned}\right\} \quad \ldots\ldots \quad (22)$$

$$\left.\begin{aligned}\Sigma\,(P\cos\beta\,x) - \Sigma\,(P\cos\alpha\,y) &= 0,\\ \Sigma\,(P\cos\gamma\,y) - \Sigma\,(P\cos\beta\,z) &= 0,\\ \Sigma\,(P\cos\alpha\,z) - \Sigma\,(P\cos\gamma\,x) &= 0.\end{aligned}\right\} \quad \ldots\ldots \quad (23)$$

Hence, the conditions of equilibrium are,

1st. *The algebraic sum of the components of the forces in the direction of any three rectangular axes must be separately equal to* 0.

2d. *The algebraic sum of the moments of the forces, with respect to any three rectangular axes, must be separately equal to* 0.

Case of a Body Restrained by a Fixed Axis.

31. If a body is restrained by a fixed axis, about which it is free to revolve, we may take this line as the axis of X. Since the axis is fixed, there can be no motion of translation, neither can there be any rotation about either of the other two axes of co-ordinates. All of Equations (22), and the first

and third of Equations (23), will be satisfied by virtue of the connection of the body with the fixed axis. The second of Equations (23) is, therefore, the only one that must be satisfied by the relation between the forces. We must have, therefore,

$$\Sigma\,(P\cos\gamma\,y - P\cos\beta\,z) = 0 \quad . \quad . \quad . \quad . \quad . \quad (24)$$

That is, if a body is restrained by a fixed axis, the forces applied to it will be in equilibrium when *the algebraic sum of the moments of the forces with respect to this axis is equal to* 0.

Quantity of Work.—Theorem of Work.

32. It was shown in Art. 9 that the work performed by a force whose line of action coincides with that of the resistance is equal to *the force* multiplied by *the space through which it acts.* It remains to consider the case in which a force acts obliquely to the resistance.

Let P be a force acting on a body O which is compelled to move along the line AB; suppose the body to move over the infinitesimal distance OC in the element of time dt. Draw Cp perpendicular to P, and PP' perpendicular to AB; denote Op by δp, the angle BOP by α, and let the quantity of work performed in the time dt be called *the elementary quantity of work.*

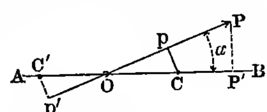

Fig. 25.

Now OP', equal to $P\cos\alpha$, is the effective component of P in the direction of AB, and the work that it performs is equal to $P\cos\alpha \times OC$, or $P \times OC\cos\alpha$; but $OC\cos\alpha$ is equal to δp, and consequently the elementary quantity of work of P is equal to $P\delta p$. Here δp falls on the force and the work is *assumed* to be positive.

If O is constrained by the action of other forces to describe

the path OC', then will δp fall on the prolongation of P and the elementary quantity of work of P will obviously be negative.

The distance δp is sometimes called the **virtual velocity** of P, and the product $P\delta p$ the **virtual moment** of P.

Let P and Q be two forces applied to a point O of a solid body and let R be their resultant; and suppose that O is constrained to move over an infinitesimal distance OC in the time dt. From C draw perpendiculars Cp to P, Cq to Q, and Cr to R, denoting Op by δp, Oq by δq, and Or by δr.

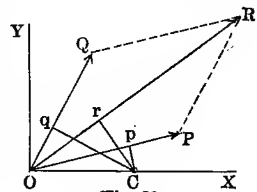

Fig. 26.

Take the line OC as the axis of X, and the perpendicular OY as the axis of Y; denote the angle XOP by α, XOQ by β, and XOR by ϕ. Then because R is the resultant of P and Q the projection of R on the axis of X is equal to the sum of the projections of P and Q on the same axis, that is,

$$R \cos \phi = P \cos \alpha + Q \cos \beta,$$

or, multiplying through by OC, we have

$$R\,(OC \cos \phi) = P\,(OC \cos \alpha) = Q\,(OC \cos \beta);$$

but, $OC \cos \phi = \delta r$, $OC \cos \alpha = \delta p$, and $OC \cos \beta = \delta q$; hence,

$$R\delta r = P\delta p + Q\delta q \quad . \; . \; . \; . \; . \quad (25)$$

Here we have supposed the path OC to be in the plane of P and Q; the result would be the same if it did not lie in that plane: for, we might then regard OC as the projection of the actual path on the plane POQ, and then from the principles of projection Op, Oq, and Or would be the projections of the path in space on the directions of P, Q, and R.

Hence, *the elementary quantity of work of the resultant of two forces is equal to the algebraic sum of*

the elementary quantities of work of the forces taken separately.

If we now regard Q as the resultant of two other forces, whose plane may or may not coincide with the plane POQ, and one of these as the resultant of two others, and so on, the principle may be extended to any number of concurrent forces, and we shall have,

$$R\delta r = \Sigma\,(P\delta p) \quad . \quad . \quad . \quad . \quad . \quad (26)$$

If we suppose the component forces applied at points invariably connected, and if we furthermore suppose that the system does not reduce to a couple, the forces will be in equilibrium when their resultant is equal to 0 (Arts. 28, 29, 30); in this case equation (26) becomes

$$\Sigma\,(P\delta p) = 0 \quad . \quad . \quad . \quad . \quad . \quad (27)$$

Hence, *if the forces applied at points invariably connected are in equilibrium, the algebraic sum of their elementary quantities of work is equal to 0.*

This is called the **theorem of work.**

III.—CENTRE OF GRAVITY AND STABILITY.

Weight and Centre of Gravity.

33. The **weight** of a body is the resultant of the weights of all its particles. The weights of the particles are sensibly directed toward the centre of the earth, and if the body be very small in comparison with the earth, they may be regarded as parallel forces; hence, the weight of a body is parallel to the weights of its particles, and is equal to their sum.

The **centre of gravity** of a body is the point through which the direction of its weight always passes. The weight being the resultant of parallel forces, the centre of gravity is a centre of parallel forces, and so long as the relative position of the particles remains unchanged, this point retains a fixed position in the body.

The position of the centre of gravity is entirely independent of the value of the intensity of gravity, provided we regard this force as constant throughout the body, which we may do in most cases. Hence, the centre of gravity is the same for the same body, wherever it may be situated. The determination of the centre of gravity is, then, reduced to the determination of the centre of a system of parallel forces.

Principles Assumed.

34. In what follows, it is assumed that the points, lines, and surfaces treated of are *material;* a **point** is then a volume whose three dimensions are *infinitesimal;* a **line** is a volume whose length is *finite,* but whose breadth and thickness are *infinitesimal;* and a **surface** is a volume whose length and breadth are *finite,* but whose thickness is *infinitesimal.*

It is also assumed that the bodies considered are **homogeneous**, that is, that the weights of different parts are proportional to their volumes; hence, the weight of a line is proportional to its *length*, the weight of a surface is proportional to its *area*, and the weight of a solid is proportional to its volume, and the reverse.

Centre of Gravity of a Straight Line.

35. Let M and N be two material points, equal in weight, and firmly connected by a line MN. The resultant of these weights will bisect the line MN in S (Art. 25); hence, S is the centre of gravity of M and N.

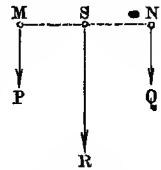

Fig. 27.

Let MN be a material straight line, and S its middle point. We may regard it as composed of material points A, A'; B, B', etc., equal in weight, and symmetrically disposed with respect to S. From what precedes, the centre of gravity of each pair of equidistant points is at S; consequently the centre of gravity of the whole line is at S. That is, *the centre of gravity of a straight line is at its middle point.*

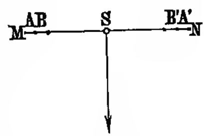

Fig. 28.

Additional Principles.

36. A **line of symmetry** of a plane figure is a straight line that bisects a system of parallel chords of the figure. If the line is perpendicular to the chords it bisects, it is a **line of right symmetry**, otherwise it is a **line of oblique symmetry**. The axes of an ellipse are lines of right symmetry; other diameters are lines of oblique symmetry.

A **plane of symmetry** of a surface, or volume, is a plane that bisects a system of parallel chords of the figure. It may be a plane of *right*, or a plane of *oblique symmetry*.

The intersection of two planes of symmetry is an **axis of symmetry.**

Let $AQBP$ be a curve, and AB a line of symmetry, bisecting the parallel chords PQ. The centre of gravity of each pair of points P, Q, is on AB (Art. 35), hence, the centre of gravity of the entire curve is on AB. Again, the centre of gravity of each chord PQ is on AB, hence the centre of gravity of the entire area is on AB. That is, *if a plane curve, or a plane area, has a line of symmetry, its centre of gravity is on that line.*

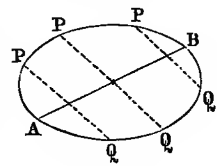

Fig. 29.

In like manner, *if a surface or volume has a plane of symmetry, its centre of gravity is in that plane.*

Two lines of symmetry, or three planes of symmetry intersecting in a point, are sufficient to determine the centre of gravity of the corresponding magnitude. Thus, all diameters of the circle are lines of symmetry, and because they intersect at the centre, it follows that the centre of gravity of both the circumference and area, is at the centre.

For a similar reason the centre of gravity of both circumference and area of an ellipse is at its centre.

Any plane through the centre of a sphere, or of an ellipsoid, is a plane of symmetry; hence the centre of gravity of either is at its centre.

The centre of gravity of any surface or volume of revolution is on its axis.

Determination of the Centre of Gravity by the Calculus.

37. Formulas (19), (20), and (21) may be adapted to the language of the calculus by changing the sign of summation, Σ, to the sign of integration, $\int$, and at the same time replacing P by dm, in which dm is the *differential* of the magni-

tude in question, and x_1, y_1, z_1, are the co-ordinates of its centre of gravity. Making these changes, we have for the co-ordinates of the centre of gravity of the magnitude, x_1, y_1, z_1, the formulas,

$$x_1 = \frac{\int x\, dm}{\int dm}; \quad . \quad . \quad . \quad . \quad . \quad (28)$$

$$y_1 = \frac{\int y\, dm}{\int dm}; \quad . \quad . \quad . \quad . \quad . \quad (29)$$

$$z_1 = \frac{\int z\, dm}{\int dm}. \quad . \quad . \quad . \quad . \quad . \quad (30)$$

In using these formulas for finding the co-ordinates of the centre of gravity we take the axes of co-ordinates so as to get the simplest results (Art. 36); we also take the integrals between such limits as to include the entire magnitude.

Centre of Gravity of a Straight Line.

38. Let AB be a straight line whose length is l. Assume the direction of AB as the axis of X, A as the origin, and let any distance from A, as AD, be denoted by x; then will dm be equal to dx. Substituting in (28) (which is the only formula required) and taking the integral between the limits 0 and l, we have,

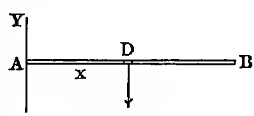

Fig. 30.

$$x_1 = \frac{\int_0^l x\, dx}{\int_0^l dx} = \tfrac{1}{2}l \quad . \quad . \quad . \quad . \quad . \quad (31)$$

Hence, *the centre of gravity is at the middle point of the line,* as was previously shown.

Centre of Gravity of a Triangle.

39. Let ABC be a plane triangle; draw a line from the vertex A to F the middle of the base BC, and take this as the axis of X; and let A be the origin. AF is a line of symmetry, and consequently it will pass through the centre of gravity. At a distance from A equal to x draw DE parallel to BC, and at an additional distance dx draw a second parallel. The area between these consecutive parallels is equal to dm. Now, if we denote BC by b and AF by h, we have, from the similar triangle ADE and ABC,

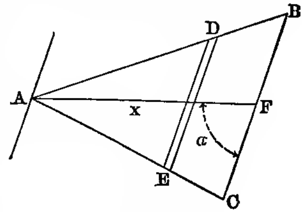

Fig. 31.

$$h : b :: x : DE, \text{ or } DE = \frac{b}{h}x.$$

The area of dm is therefore equal to

$$\frac{b}{h}x \times dx \sin \alpha,$$

α being the angle AFC. Substituting this in (28), dropping the constant factor, and integrating between the limits 0 and h, we have

$$x_1 = \frac{\int_0^h x^2\,dx}{\int_0^h x\,dx} = \tfrac{2}{3}h \quad . \quad . \quad . \quad . \quad . \quad (32)$$

Hence, *the centre of gravity of a triangle is on a line drawn from the vertex to the middle of the base and two thirds of the distance from the vertex to the base.*

Centre of Gravity of a Pyramid or Cone.

40. Let $ABCDE$ be any pyramid and let AG be a line from its vertex to the centre of gravity of its base. Take AG as the axis of X, and denote the area of the base by b; also denote AG by h.

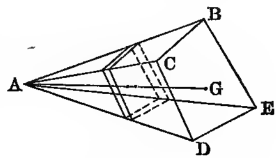

Fig. 32.

Now, we may regard the pyramid as made up of slices formed by planes parallel to the base, and whose distances apart, estimated along AG, are equal to dx; the actual thickness of each slice is $dx \times \sin\alpha$, α being the angle between AG and the base. It is obvious from the principles of Geometry that AG will pass through the centre of gravity of each slice; hence, the centre of gravity of the pyramid will be on AG.

At a distance from A denoted by x pass a plane parallel to the base, and at an additional distance dx pass a second parallel plane; the volume included by these planes will be dm. Denoting the section of the first of these planes by b', we shall have

$$b : b' :: h^2 : x^2, \text{ or } b' = \frac{b}{h}x^2.$$

Hence, the magnitude of the corresponding slice is equal to $\frac{b}{h}x^2\,dx \sin\alpha$. Substituting this for dm in (28), striking out the common constant factor, and integrating from 0 to h, we have

$$x_1 = \frac{\int_0^h x^3\,dx}{\int_0^h x^2\,dx} = \tfrac{3}{4}h \quad . \quad . \quad . \quad . \quad . \quad (33)$$

Hence, *the centre of gravity of a pyramid is on a*

line drawn from the vertex to the centre of gravity of the base, and at three fourths of the distance from the vertex to the base.

The demonstration is applicable to any pyramid or cone.

Centre of Gravity of an Arc of a Circle.

41. Let ABC be an arc whose radius is r, and let the origin be at the centre O. Take the axis of X perpendicular to the chord AC, in which case it will be a line of symmetry. Denote the semi-angle of the arc by α and any variable angle, BOK, by θ. Now, if we give to θ the constant increment $d\theta$, the differential of the arc, or dm, will be equal to $rd\theta$ and its lever arm OL, or x, will be equal to $r\cos\theta$. Substituting in (28) and integrating from $\theta = -\alpha$ to $\theta = +\alpha$, we have

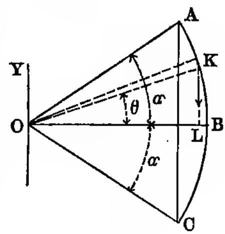

Fig. 33.

$$x_1 = \frac{\int_{-\alpha}^{\alpha} r^2 \cos\theta\, d\theta}{\int_{-\alpha}^{\alpha} r\, d\theta} = \frac{2r^2 \sin\alpha}{2r\alpha} \quad . \; . \; . \; . \; . \quad (34)$$

But $2r\sin\alpha = AC$, and $2r\alpha = \text{arc } ABC$; hence,

$$x_1 = \frac{r \times AC}{\text{arc } ABC}, \text{ or arc } ABC : AC :: r : x_1.$$

That is, *the centre of gravity of an arc of a circle is on the diameter which bisects its chord, and its distance from the centre is a fourth proportional to the arc, chord, and radius.*

When $\alpha = \frac{1}{2}\pi$, or 90°, the arc becomes a semi-circumference, $\sin\alpha = 1$, and equation (34) gives

$$x_1 = \frac{2r}{\pi}.$$

Centre of Gravity of one Branch of a Cycloidal Arc.

42. Let ODX be one branch of a cycloid whose base is OX and whose axis is CD. Let the axis of X coincide with the base and let the axis of Y be parallel to the axis of the curve; then, (Calculus, p. 100), will CD be a line of symmetry. Any element K of the curve will be equal to $\sqrt{dx^2 + dy^2}$, or substituting for dx, its value (Calc., p. 100) and reducing, we have

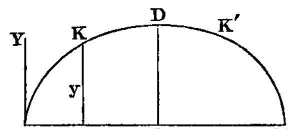

Fig. 34.

$$dm = \sqrt{2r}\,(2r - y)^{-\frac{1}{2}}\,dy.$$

Substituting in (29), suppressing the common constant factor and integrating from $y = 0$ to $y = 2r$, we have

$$y_1 = \frac{\int_0^{2r} (2r - y)^{-\frac{1}{2}} y\,dy}{\int_0^{2r} (2r - y)^{-\frac{1}{2}}\,dy} = \tfrac{4}{3}r = \tfrac{2}{3}\,(2r) \quad . \; . \; . \; . \; . \quad (35)$$

NOTE.—The numerator is reduced by Formula A, and the integration is completed by formula [29] Calculus, giving

$$N = \tfrac{2}{3}\,(2r - y)^{\frac{1}{2}} y - \tfrac{8}{3}\,r\,(2r - y)^{\frac{1}{2}}.$$

The denominator is integrated by formula [29], giving

$$D = -\,2\,(2r - y)^{\frac{1}{2}}.$$

For $y = 0$, $N = -\,\frac{8}{3}r\sqrt{2r}$, and $D = -\,2\sqrt{2r}$; for $y = 2r$ both N and D are 0: hence, the result given in (35).

The value of y_1 is the distance of the centre of gravity of OKD from the axis of X, but by reason of symmetry it is also the distance of the centre of gravity of $XK'D$ from the same axis.

Hence, *the centre of gravity of the entire curve is on the axis of the curve and at a distance from the base equal to* $\frac{2}{3}CD$.

Centre of Gravity of a Sector of a Circle.

43. Let AOC be the sector; let the axis of X bisect it and assume the same notation as in Art. 41. In this case dm is equal to the elementary sector OKL and its value is $\frac{1}{2}r^2d\theta$: because this sector may be regarded as a triangle, its centre of gravity is two thirds of the distance from O to K, and consequently its lever arm x is equal to $\frac{2}{3}r\cos\theta$. Substituting in (28), suppressing the common constant factor, and integrating from $-\alpha$ to $+\alpha$, we have

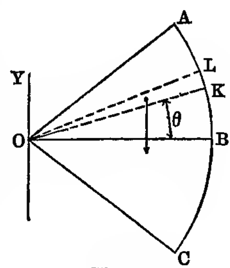

Fig. 35.

$$x_1 = \frac{2}{3}\frac{\int_{-\alpha}^{\alpha} r\cos\theta}{\int_{-\alpha}^{\alpha} d\theta} = \frac{2}{3}r\frac{\sin\alpha}{\alpha} \quad . \quad . \quad . \quad . \quad . \quad (36)$$

If $\alpha = \frac{1}{2}\pi$ (or 90°), the sector becomes a semicircle, $\sin\alpha = 1$, and we have

$$x_1 = \frac{4r}{3\pi}.$$

If we make dm equal to the second differential of a magnitude, the position of the centre of gravity is found by *double integration.*

Centre of Gravity of a Sector of a Circular Ring.

44. A circular ring is bounded by two arcs of concentric circles and by two radii, as $BCC'B'$. Take the origin at the centre of the circle; denote the radius of the outer circle by

r, that of the inner circle by r', the angle XOC by α' and the angle XOB by α.

Draw any radius between r and r', as OL, and denote the angle which it makes with the axis of X by θ. If we increase θ by $d\theta$, and draw the corresponding radius OL', the part of the ring between these radii will be its *first differential*. Now take any distance OK greater than r' and less than r and denote it by z: describe two arcs, one with the radius z and the other with the radius $z + dz$; the part of the first differential included between these arcs will be the *second differential* of $BCC'B'$. Regarding this as a rectangle, its length is $zd\theta$ and its breadth is dz and its area, d^2m, is equal to $zdzd\theta$; furthermore, its lever arm x is equal to $z \cos \theta$. Substituting these values for dm and x in (28), and taking the double integral between the limits indicated, we have

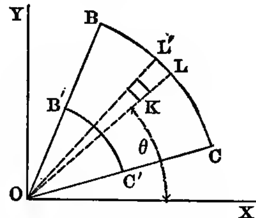

Fig. 36.

$$x_1 = \frac{\int_{\alpha'}^{\alpha}\int_{r'}^{r} z^2 dz \cos\theta\, d\theta}{\int_{\alpha'}^{\alpha}\int_{r'}^{r} z dz\, d\theta} = \frac{2}{3}\frac{r^3 - r'^3}{r^2 - r'^2}\frac{\int_{\alpha'}^{\alpha} \cos\theta\, d\theta}{\int_{\alpha'}^{\alpha} d\theta}$$

$$= \frac{2}{3}\frac{r^3 - r'^3}{r^2 - r'^2} \cdot \frac{\sin\alpha - \sin\alpha'}{\alpha - \alpha'} \quad . \quad . \quad . \quad . \quad (37)$$

Because the centre of gravity is on the bisecting line of the sector, the value of x_1 is sufficient to determine it. If we make $\alpha' = -\frac{1}{2}\pi$, $\alpha = \frac{1}{2}\pi$ and $r' = o$, we have as before,

$$x_1 = \frac{4r}{3\pi}.$$

Centre of Gravity of a Semi-Ellipsoid.

45. Let the axis of X be taken to coincide with the axis of the ellipsoid, the origin being at the centre, and suppose

the volume to be generated by revolving a semi-ellipse around its transverse axis. The elementary volume will be a slice perpendicular to the axis of X, whose radius is y and whose thickness is dx. The value of y^2 taken from the equation of the ellipse is $\frac{b^2}{a^2}(a^2 - x^2)$; hence,

$$dm = \pi y^2 dx = \pi \frac{b^2}{a^2} (a^2 - x^2)\, dx.$$

Substituting this in (28), striking out the common constant factor, and integrating from $x = 0$ to $x = a$, we have,

$$x_1 = \frac{\int_0^a (a^2 - x^2)\, x\, dx}{\int_0^a (a^2 - x^2)\, dx} = \tfrac{3}{8}a \quad . \quad . \quad . \quad . \quad . \quad (38)$$

That is, *the centre of gravity of a semi-prolate spheroid is on its axis and at a distance from the centre equal to three-eighths of the transverse axis of the meridian curve.*

Centre of Gravity of a Trirectangular Spherical Pyramid.

46. Let O be the centre of sphere whose radius is r, and let AOC, AOB, and BOC be three rectangular planes. The volume bounded by these planes and the trirectangular spherical triangle, ABC, is the pyramid in question. Let AOD be any angle ϕ, and through OD pass a plane, DOC, cutting the base of the pyramid in the quadrant, CD; draw OE, making the angle DOE equal to $d\phi$, and through OE pass

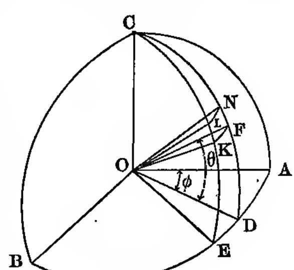

Fig. 37.

the plane EOC; the semi-wedge between these planes is the *first differential* of the pyramid.

Draw a radius, OF, making the angle DOF equal to θ, and also the consecutive radius ON, making FON equal to $d\theta$; then draw arcs of small circles through F and N, cutting the arc CE in K and L; the elementary pyramid determined by the radii to F, N, K, and L, is the *second differential* of the pyramid.

The arc FK is equal to $ED \cos\theta = rd\phi \times \cos\theta$, and NF is equal to $rd\theta$; hence, the area of the base of the elementary pyramid, regarded as a rectangle, is $r^2 \cos\theta \, d\theta \, d\phi$, and because its altitude is r, the volume of the pyramid is given by the equation,

$$d^2m = \tfrac{1}{3} r^3 \cos\theta \, d\theta \, d\phi.$$

The centre of gravity of the pyramid is at a distance from O equal to three fourths of r; hence, the co-ordinate x of this point is given by the equation,

$$x = \tfrac{3}{4} r \cos\theta \cos\phi.$$

Substituting these values in (28), striking out the common constant factor, and taking the double integral between the limits of the figure, we have,

$$x_1 = \tfrac{3}{4} \frac{\int_0^{90^\circ}\int_0^{90^\circ} r \cos^2\theta \, d\theta \cos\phi \, d\phi}{\int_0^{90^\circ}\int_0^{90^\circ} \cos\theta \, d\theta \, d\phi} = \tfrac{3}{8} r \quad . \quad . \quad . \quad . \quad . \quad (39)$$

Note.—From example 8, p. 151, Calculus, we have at once,

$$\int \cos^2\theta \, d\theta = \tfrac{1}{2} (\cos\theta \sin\theta + \theta).$$

From the principle of symmetry we see that y_1 and z_1 are each equal to $\frac{3}{8} r$. The centre of gravity is on the radius that is equally inclined to OA, OB, and OC, and its distance from O is equal to $\sqrt{x_1^2 + y_1^2 + z_1^2}$, or to $\frac{3}{8} r \sqrt{3}$.

The Centrobaric Method.

47. If we clear equation (29) of fractions, and multiply both members by 2π, we have,

$$2\pi y_1 \int dm = \int 2\pi y dm \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

First, let m be a plane curve; then will dm be equal to $\sqrt{dx^2 + dy^2}$, and (a) becomes, after limiting the curve by the ordinates a and b,

$$2\pi y_1 \int_a^b (dx^2 + dy^2)^{\frac{1}{2}} = \int_a^b 2\pi y \,(dx^2 + dy^2)^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad (40)$$

Referring to the Calculus, pp. 91–92, we see that the first member of this equation is $2\pi y_1$ *multiplied* by the length of a definite portion of a plane curve; and that the second member is the surface generated by revolving that portion of the curve around the axis of X.

Hence, *the area of a surface of revolution is equal to the length of its generating arc multiplied by the circumference described by its centre of gravity.*

Secondly, let m be a plane area; then will $dm = ydx$, and the ydm of (29) will equal $\frac{1}{2}y \times ydx$, and we shall have, as before,

$$2\pi y_1 \int_a^b ydx = \int_a^b \pi y^2 dx \quad . \quad . \quad . \quad . \quad . \quad (41)$$

The first member of this equation is $2\pi y_1$ *multiplied* by a definite plane area, and the second member is the volume generated by revolving this area around the axis of X.

Hence, *the volume of a solid of revolution is equal to its generating area multiplied by the circumference, described by its centre of gravity.*

These principles may be used for finding the measure of

surfaces and solids of revolution, or, in certain cases, for finding the centres of gravity of their generating magnitudes. We are only concerned with the latter.

To illustrate the manner of applying the rules just deduced, let us take the following examples :

1°. *Centre of gravity of a semicircular arc.*—The surface generated by revolving it about its diameter is $4\pi r^2$, and the length of the semicircle is πr ; hence, from (40),

$$4\pi r^2 = 2\pi y_1 \times \pi r, \quad \text{or} \quad y_1 = \frac{2r}{\pi} \quad . \quad . \quad . \quad . \quad . \quad (42)$$

2°. *Centre of gravity of a cycloidal arc.*—The surface generated by revolving one branch of the arc around its base is (Calc., p. 167), $\frac{64}{3}\pi r^2$, and the length of the branch is (Calc., p. 158), $8r$; hence, from (40),

$$\frac{64}{3}\pi r^2 = 2\pi y_1 \times 8r, \quad \text{or} \quad y_1 = \frac{4}{3}r = \frac{2}{3}(2r) \quad . \quad . \quad . \quad . \quad . \quad (43)$$

3°. *Centre of gravity of the area of a semicircle.*—The volume generated by revolving it around its diameter is $\frac{4}{3}\pi r^3$, and the area of the semicircle is $\frac{1}{2}\pi r^2$; hence, from (40),

$$\frac{4}{3}\pi r^3 = 2\pi y_1 \times \frac{1}{2}\pi r^2, \quad \text{or} \quad y_1 = \frac{4r}{3\pi} \quad . \quad . \quad . \quad . \quad . \quad (44)$$

4°. *Centre of gravity of the area of one branch of a cycloid.*—The volume generated by revolving it around its base is (Calc., p. 170), $5\pi^2 r^3$, and the area of the branch is (Calc., p. 163), $3\pi r^2$; hence, from (40),

$$5\pi^2 r^3 = 2\pi y_1 \times 3\pi r^2, \quad \text{or} \quad y_1 = \frac{5}{6}r = \frac{5}{12}(2r).$$

NOTE.—The centre of gravity is on the line of symmetry in each case.

Experimental Determination of the Centre of Gravity.

48. In certain cases the approximate position of the centre of gravity of a body may be found with sufficient accuracy as follows: Attach a string at any point C, and suspend the body by it; when the body comes to rest, mark the direction of the string; then suspend the body by a second point, B, and when it comes to rest, mark the direction of the string; the point of intersection, G, will be the centre of gravity of the body.

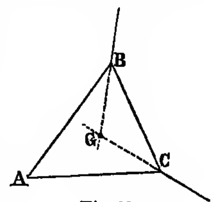

Fig. 38.

Instead of suspending the body by a string, it may be balanced on a point. In this case, the weight acts vertically downward, and is resisted by the reaction of the point; hence, the centre of gravity lies vertically over the point.

If, therefore, a body be balanced at any two points of its surface, and verticals be drawn through the points in these positions, their intersection will be the centre of gravity of the body.

Centre of Gravity by Composition.

49. When we know the centres of gravity of two or more bodies, or *parts* of the same body, and the weights of each, the centre of gravity of the whole may be found by the method of composition; thus, if A and B are the centres of gravity of two bodies whose weights are W and W', the weights being parallel forces, their resultant, $W + W'$, will be applied at that point of AB which divides it into segments that are inversely as the forces. If, for example, the weight of A is 9 *lbs.*, and that of B 7 *lbs.*, we divide AB into 16 equal parts, and lay off 7 of these from A towards B, which will determine the centre of gravity of A and B.

If the two bodies are homogeneous, we may use their magnitudes instead of their weights.

Having found the common centre of gravity of A and B, we may compound it with C in the same manner, and so on till all the bodies have been considered.

As an example, let it be required to find the centre of gravity of a polygon as A, B, C, D, E. Divide it into triangles, and find the centre of gravity of each triangle. The weights of these triangles are proportional to their areas, and may be represented by them. Let O, O', O'', be the centres of gravity of the triangles into which the polygon is divided. Join O and O', and find a point O''', such that

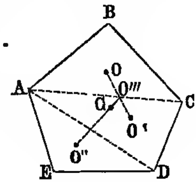

Fig. 39.

$$O'\,O''' : OO''' :: ABC : ACD;$$

then will O''' be the centre of gravity of the triangles ABC and ACD.

Join O'' and O''', and find a point, G, such that

$$O'''G : O''G :: ADE : ABC + ACD;$$

then will G be the centre of gravity of the polygon.

To find the centre of gravity of a polyhedron; if we take a point within and join it with each vertex of the polyhedron, we shall form as many pyramids as the solid has faces; the centre of gravity of each pyramid may be found by the rule. If the centres of gravity of the first and second pyramid be joined by a straight line, the common centre of gravity of the two may be found by a process similar to that used in finding the centre of gravity of a polygon, observing that the weights of the pyramids are proportional to their volumes, and may be represented by them. Having compounded the weights

of the first and second, and found its point of application, we may, in like manner, compound the weight of these two with that of the third, and so on; the last point of application will be the centre of gravity of the polyhedron.

Composition by the Method of Moments.

50. When we have several bodies, and it is required to find their common centre of gravity, it will often be found convenient to employ the principle of moments. To do this, we first find the centre of gravity of each body separately, by rules already given. The weight of each body is then regarded as a force, applied at the centre of gravity of the body. The weights being parallel, we have a system of parallel forces, whose points of application are known. If these points are all in the same plane, we find the lever arms of the resultant of all the weights, with respect to two lines, at right angles to each other in that plane; and these will make known the point of application of the resultant, or, what is the same thing, the centre of gravity of the system. If the points are not in the same plane, the lever arms of the resultant are found, with respect to three axes, at right angles to each other; these make known the point of application of the resultant weight, or the position of the centre of gravity.

The methods of applying this and the preceding article are illustrated by the following:

Examples.

1. Required the point of application of the resultant of three equal weights, applied at the vertices of a plane triangle.

Solution.—Let A, B, and C be the vertices of the triangle and D the middle of BC. The resultant of the weights at B and C will be applied at D. The resultant of the three weights will be applied at a point G of AD, and because the weight at D is twice that at A the distance AG is

two thirds of AD, that is, the required point is at the centre of gravity of the triangle.

2. Required the point of application of the resultant of a system of equal parallel forces, applied at the vertices of a regular polygon.

Ans. At the centre of the polygon.

3. Parallel forces of 3, 4, 5, and 6 lbs. are applied at the successive vertices of a square, whose side is 12 inches. At what distance from the first vertex is the point of application of their resultant?

SOLUTION.—Take the sides of the square through the first vertex as axes; call the side through the first and second vertex, the axis of X, and that through the first and fourth, the axis of Y. We shall have, from Formulas (28, 29),

$$x_1 = \frac{4 \times 12 + 5 \times 12}{18} = 6;$$

and

$$y_1 = \frac{6 \times 12 + 5 \times 12}{18} = \frac{22}{3}.$$

Denoting the required distance by d, we have

$$d = \sqrt{x_1^2 + y_1^2} = 9.475 \text{ in.} \quad \textit{Ans.}$$

4. Seven equal forces are applied at seven of the vertices of a cube. What is the distance of the point of application of their resultant from the eighth vertex?

SOLUTION.—Take the eighth vertex as the origin of co-ordinates, and the three edges passing through it as axes. We shall have, from Equations (28, 29, 30), denoting one edge of the cube by a,

$$x_1 = \tfrac{4}{7}a, \; y_1 = \tfrac{4}{7}a, \quad \text{and} \quad z_1 = \tfrac{4}{7}a.$$

Denoting the required distance by d, we have

$$d = \sqrt{x_1^2 + y_1^2 + z_1^2} = \tfrac{4}{7}a\sqrt{3}. \quad \textit{Ans.}$$

5. Two isosceles triangles are constructed on opposite sides of the base b, having altitudes equal to h and h', h being greater than h'. Where is the centre of gravity of the space within the two triangles?

SOLUTION.—It must lie on the altitude of the greater triangle. Take the common base as an axis of moments; then will the moments of the triangles be $\frac{1}{2}bh \times \frac{1}{3}h$, and $\frac{1}{2}bh' \times \frac{1}{3}h'$; and we have

$$x_1 = \frac{\frac{1}{6}b(h^2 - h'^2)}{\frac{1}{2}b(h + h')} = \tfrac{1}{3}(h - h').$$

That is, the centre of gravity is on the altitude of the greater triangle, at a distance from the base equal to one third of the difference of the two altitudes.

6. Where is the centre of gravity of the space between two circles tangent to each other internally?

SOLUTION.—Take their common tangent as an axis of moments. The centre of gravity will lie on the common normal, and its distance from the point of contact is given by the equation,

$$x_1 = \frac{\pi r^3 - \pi r'^3}{\pi r^2 - \pi r'^2} = \frac{r^2 + rr' + r'^2}{r + r'}.$$

7. Let there be a square, divided by its diagonals into four equal parts, one of which is removed. Required the distance of the centre of gravity of the remaining figure from the opposite side of the square.

Ans. $\frac{7}{18}$ of the side of the square.

8. To construct a triangle, having given its base and centre of gravity.

SOLUTION.—Draw through the middle of the base, and the centre of gravity, a straight line; lay off beyond the centre of gravity a distance equal to twice the distance from the middle of the base to the centre of gravity. The point thus found is the vertex.

9. Three men carry a cylindrical bar, one taking hold of one end, and the others at a common point. Required the position of this point, in order that the three may sustain equal portions of the weight.

Ans. At three-fourths the length of the cylinder from the first.

STABILITY AND EQUILIBRIUM.

Stable, Unstable, and Indifferent Equilibrium.

51. A body is in *stable* equilibrium when, on being slightly disturbed from a state of rest, it has a tendency to return to that state. This will be the case when the centre of gravity of the body is at its lowest point. Let A be a body suspended from an axis O, about which it is free to turn. When the centre of gravity of A lies vertically below the axis, it is in equilibrium, for the weight of the body is exactly counterbalanced by the resistance of the axis.

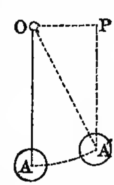

Fig. 40.

Moreover, the equilibrium is *stable;* for if the body be deflected to A', its weight acts with the lever arm OP to restore it to its position of rest, A.

A body is in *unstable* equilibrium when, on being slightly disturbed from its state of rest, it tends to depart still farther from it. This will be the case when the centre of gravity of the body occupies its highest position.

Let A be a sphere, connected by an inflexible rod with the axis O. When the centre of gravity of A is vertically above O, it is in *unstable* equilibrium; for, if the sphere be deflected to the position A', its weight will act with the lever arm OP to increase the deflection. The motion continues till, after a few vibrations, it comes to rest below the axis. In this last position, it is in stable equilibrium.

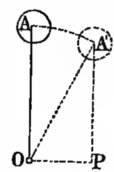

Fig. 41.

A body is in *indifferent,* or *neutral,* equilibrium when it remains at rest, wherever it may be placed. This is the case when the centre of gravity continues in the same horizontal plane on being slightly disturbed.

Let A be a sphere, supported by a horizontal axis OP through its centre of gravity. Then, in whatever position it may be placed, it will have no tendency to change this position; it is, therefore, in *indifferent,* or *neutral* equilibrium.

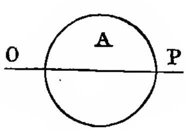

Fig. 42.

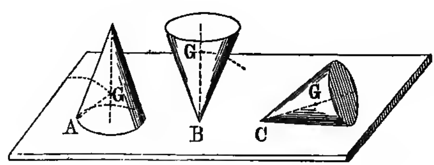

Fig. 43.

In figure 43, A, B, and C, represent a cone in positions of *stable, unstable,* and *indifferent* equilibrium.

If a wheel be mounted on a horizontal axis, about which it is free to turn, the centre of gravity not lying on the axis, it

will be in stable equilibrium, when the centre of gravity is directly below the axis; and in unstable equilibrium when it is directly above the axis. When the axis passes through the centre of gravity, it will, in every position, be in neutral equilibrium.

We infer, from the preceding discussion, that when a body at rest is so situated that it cannot be disturbed from its position without raising its centre of gravity, it is in a state of *stable equilibrium;* when a slight disturbance depresses the centre of gravity, it is in a state of *unstable equilibrium;* when the centre of gravity remains constantly in the same horizontal plane, it is in a state of *neutral equilibrium.*

This principle holds true in the combinations of wheels and other pieces used in machinery, and indicates the importance of balancing these elements, so that their centres of gravity may remain in the same horizontal planes.

Stability of Bodies on a Horizontal Plane.

52. A body resting on a horizontal plane may touch it in one, or in more than one point. In the latter case, the salient polygon, formed by joining the extreme points of contact, as *abcd,* is called the *polygon of support.*

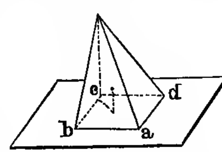

Fig. 44.

When the direction of the weight of the body, that is, the vertical through its centre of gravity, pierces the plane within the polygon of support, the body is *stable,* and will remain in equilibrium, unless acted upon by some other force than the weight of the body. In this case, the body will be most easily overturned about that side of the polygon of support which is nearest to the line of direction of the weight. The moment of the weight, with respect to this side, is called the *moment of stability.*

Denoting the weight of the body by W, the distance from

its line of direction to the nearest side of the polygon of support by r, and the moment of stability by S, we have,

$$S = Wr.$$

The moment of stability is the moment of the least extraneous force that is capable of overturning the body. The weight of a body remaining the same, its *stability* increases with r. If the polygon of support is a regular polygon, the stability will be greatest, other things being equal, when the direction of the weight passes through its centre. The area of the polygon of support remaining constant, the stability will be greater as the polygon approaches a circle.

When the direction of the weight passes without the polygon of support, the body is *unstable,* and unless supported by some other force than the weight, it will turn about the side nearest the direction of the weight. In this case, the product of the weight into the distance from its direction to the nearest side of the polygon, is called *the moment of instability.*

The moment of instability is equal to the least moment of a force that can prevent the body from overturning.

If the direction of the weight intersect any side of the polygon of support, the body will be *in a state of equilibrium bordering on rotation about that side.*

In what precedes, we supposed the supporting plane to be horizontal, and that the only force acting is the weight of the body.

Stability when One Body Presses on Another.

53. When other forces than the weight are acting on a body, and when the plane of contact of the two bodies has any position, the conditions of stability require, *first,* that the resultant of all the forces shall pass within the polygon of

support, to prevent rotation; and, *secondly,* that it shall be normal to the plane of contact, to prevent sliding.

For example, let A be a movable body pressed against a fixed body B, and touching it at a single point, P'. In order that A may be in equilibrium, the resultant of all the forces acting on it, including its weight, must pass through the point of contact, P'; otherwise there would be a tendency to rotation about P', which would be measured by the moment of the resultant with respect to this point. Furthermore, the direction of the resultant must be normal to the surface of B at the point P', else the body A would have a tendency to slide along the body B, which tendency would be measured by the tangential component. The pressure on B develops a force of reaction, which is equal and directly opposed to it.

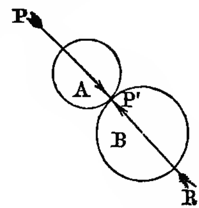

Fig. 45.

Practical Problems.

1. A horizontal beam AB, which sustains a load, is supported by a pivot at A, and by a cord DE, the point E being vertically over A Required the tension of DE, and the vertical pressure on A.

Solution.—Denote the weight of the beam and load by W, and suppose its point of application to be C. Denote CA by p, the perpendicular distance, AF, from A to DE, by p', and the tension of the cord by t. If we take A as a centre of moments, we have, in case of equilibrium,

$$Wp = tp'; \qquad \therefore t = W\frac{p}{p'}.$$

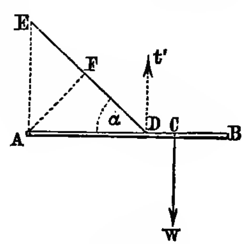

Fig. 46.

Or, denoting the angle EDA by a, and the distance AD by b, we have,

$$p' = b \sin a; \qquad \therefore t = W\frac{p}{b \sin a}.$$

To find the vertical pressure on A, resolve t into components, parallel and perpendicular to AB. We have for the latter component, denoted by t',

$$t' = t \sin \alpha = W\frac{p}{b}.$$

The vertical pressure on A, plus the weight W, must be equal to t'. Denoting the vertical pressure by P, we have,

$$P + W = W\frac{p}{b}; \text{ or, } P = W\left(\frac{p}{b} - 1\right) = W\left(\frac{p-b}{b}\right);$$

or,
$$P = W\frac{DC}{AD}.$$

When $DC = 0$; or, when D and C coincide, the vertical pressure is 0.

2. A rope, AD, supports a pole, DO, one end of which rests on a horizontal plane, and from the other is suspended a weight W. Required the tension of the rope, and the thrust, or pressure, on the pole, its weight being neglected.

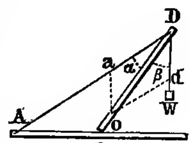

Fig. 47.

Solution.—Denote the tension of the rope by t, the pressure on the pole by p, the angle ADO by α, and the angle ODW by β.

There are three forces acting at D, which hold each other in equilibrium; the weight W, acting downward, the tension of the rope, acting from D, toward A, and the reaction of the pole, acting from O toward D. Lay off Dd, to represent the weight, and complete the parallelogram $daoD$; then will Da represent the tension of the rope, and Do the thrust on the pole.

From Eq. 16, we have,

$$t : W :: \sin\beta : \sin\alpha; \qquad \therefore t = W\frac{\sin\beta}{\sin\alpha}.$$

We have, also, from the same principle,

$$p : W :: \sin(\alpha + \beta) : \sin\alpha; \qquad \therefore p = W\frac{\sin(\alpha+\beta)}{\sin\alpha}.$$

If the rope is horizontal, we have $\alpha = 90° - \beta$, which gives,

$$t = W\tan\beta, \text{ and } p = \frac{W}{\cos\beta}.$$

3. A beam FB, is suspended by ropes attached at its extremities, and fastened to pins A and H. Required the tensions of the ropes.

Solution.—Denote the weight of the beam and its load by W, and let c be its point of application. Denote the tension of the rope BH, by t, and that of FA by t'. The forces in equilibrium, are W, t and t'. The plane of these forces must be vertical, and further, the directions of the forces must intersect in a point. Produce AF and BH, till they intersect in K, and draw Kc; take Kc, to represent the weight of the beam and its load, and complete the parallelogram $Kbcf$; then will Kb represent t, and Kf will represent t'. Denote the angle cKB by α, and cKF by β. We shall have, as in the last problem,

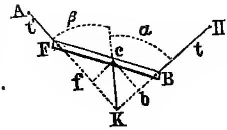

Fig. 48.

$$W : t :: \sin(\alpha + \beta) : \sin\beta; \qquad \therefore\ t = W\frac{\sin\beta}{\sin(\alpha+\beta)}.$$

And,

$$W : t' :: \sin(\alpha + \beta) : \sin\alpha; \qquad \therefore\ t' = W\frac{\sin\alpha}{\sin(\alpha+\beta)}.$$

4. A gate AH, is supported at O on a pivot, and at A by a hinge, attached to a post AB. Required the pressure on the pivot, and the tension of the hinge.

Solution.—Denote the weight of the gate and its load by W, and let C be its point of application. Produce the vertical through C, till it intersects the horizontal through A in D, and draw DO. Then will AD and DO be the directions of the required components of W. Lay off Dc, to represent W, and complete the parallelogram, $Dcoa$; then will Do represent the pressure on O, and aD the tension on the hinge, A. Denoting the angle oDc by α, the pressure on the pivot by p, and on the hinge by p', we have,

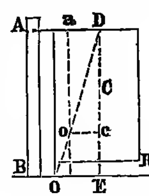

Fig. 49.

$$p = \frac{W}{\cos\alpha}, \quad \text{and} \quad p' = p\sin\alpha.$$

If we denote OE by b, and DE by h, we shall have,

$$\cos\alpha = \frac{h}{\sqrt{b^2+h^2}}, \quad \text{and} \quad \sin\alpha = \frac{b}{\sqrt{b^2+h^2}}.$$

Hence,

$$p = \frac{W\sqrt{b^2+h^2}}{h}, \quad \text{and} \quad p' = \frac{pb}{\sqrt{b^2+h^2}}.$$

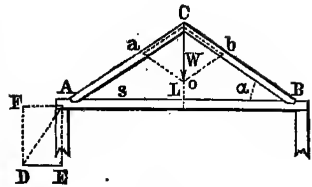

Fig. 50.

5. Having two rafters, AC and BC, abutting in notches of a tie-beam AB, it is required to find the pressure, or *thrust*, on the rafters, and the direction and intensity of the pressure on the joints at the tie-beam.

Solution.—Denote the weight of the rafters and their load by $2w$; we may regard this weight as made up of three parts—a weight w, applied at C, and two equal weights $\frac{1}{2}w$, applied at A and B respectively. Denote the half span AL by s, the rise CL by h, and the length of the rafter AC or CB by l. Denote, also, the angle CBL by a, the thrust on each rafter by t, and the resultant pressure at each of the joints A and B by p.

Lay off Co to represent the weight w, and complete the parallelogram $Cboa$; then will Ca and Cb represent the thrust on the rafters; and, since $Cboa$ is a rhombus, we have,

$$t \sin a = \tfrac{1}{2}w \qquad \therefore\ t = \frac{w}{2 \sin a} = \frac{wl}{2h}.$$

Conceive t to be applied at A, and there resolve it into components parallel to CL and LA; we have, for these components,

$$t \sin a = \tfrac{1}{2}w, \quad \text{and} \quad t \cos a = \frac{ws}{2h}.$$

The latter component gives the strain on the tie-beam, AB.

To find the pressure on the joint, we have, acting downward, the forces $\frac{1}{2}w$ and $\frac{1}{2}w$, or the single force w, and, acting from L toward A, the force $\frac{ws}{2h}$; hence,

$$p = \sqrt{w^2 + \frac{w^2s^2}{4h^2}} = \frac{w}{2h}\sqrt{4h^2 + s^2}.$$

If we denote the angle DAE by β, we have from the right-angled triangle DAE,

$$\tan \beta = \frac{DE}{AE} = \frac{ws}{2h} \div w = \frac{s}{2h}.$$

The joint should be perpendicular to the force p, that is, it should make with the horizon an angle whose tangent is $\frac{s}{2h}$.

6. In the last problem suppose the rafters to abut against the wall. Required the least thickness that must be given to it to prevent it from being overturned.

Solution.—Denote the weight thrown on the wall by w, the length of wall that sustains the pressure p by l', its height by h', its thickness by x, and the weight of each cubic foot of wall by w'; then will the weight of this part be $w'h'l'x$.

The force $\frac{ws}{2h}$ acts with an arm of lever h' to overturn the wall about its lower and outer edge; this force is resisted by the weight $w+w'h\,l'x$, acting through the centre of gravity of the wall with a lever arm equal to $\frac{1}{2}x$. If there be an equilibrium, the moments of these two forces are equal, that is,

$$\frac{ws}{2h} \times h' = (w + w'h'l'x)\frac{x}{2}, \text{ or } \frac{wsh'}{h} = wx + w'h'l'x^2.$$

Reducing, we have, $x^2 + \frac{w}{w'h'l'}x = \frac{ws}{w'l'h}$;

$$\text{or, } x = -\frac{w}{2w'h'l'} \pm \sqrt{\frac{ws}{w'hl'} + \frac{w^2}{4w'^2h'^2l'^2}}.$$

7. A sustaining wall has a cross section in the form of a trapezoid, the face on which the pressure is thrown being vertical, and the opposite face having a slope of *six* perpendicular to *one* horizontal. Required the least thickness that must be given to the wall at top, that it may not be overturned by a horizontal pressure, whose point of application is at a distance from the bottom of the wall equal to one third its height.

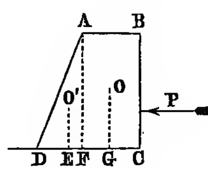

Fig. 51.

Solution.—Pass a plane through the edge A parallel to BC, and consider a portion of the wall whose length is one foot. Denote the pressure on this by P, the height of the wall by $6h$, its thickness at top by x, and the weight of a cubic foot by w. Let fall from the centres of gravity O and O' of the two portions, perpendiculars OG and $O'E$, and take the edge D as an axis of moments. The weight of the portion $ABCF$ is equal to $6whx$, and its lever arm, DG, is equal to $h + \frac{1}{2}x$. The weight of the portion ADF is $3wh^2$, and its lever arm, DE, is $\frac{2}{3}h$. In case of equilibrium,

the sum of the moments of their weights must be equal to the moment of P, whose lever arm is $2h$. Hence,

$$6whx(h + \tfrac{1}{2}x) + 3wh^2 \times \tfrac{2}{3}h = P \times 2h\,;$$

or,

$$6whx + 3wx^2 + 2wh^2 = 2P.$$

Whence,

$$x^2 + 2hx = \frac{2(P - wh^2)}{3w};$$

$$\therefore\ x = -\,h \pm \sqrt{\frac{2(P - wh^2)}{3w} + h^2}.$$

8. Required the conditions of stability of a square pillar acted on by a force oblique to the axis, and applied at the centre of gravity of the upper base.

SOLUTION.—Denote the intensity of the force by P, its inclination to the vertical by a, the breadth of the pillar by $2a$, its height by x, and its weight by W. Through the centre of gravity of the pillar draw a vertical AC, and lay off AC equal to W; prolong PA and lay off AB equal to P; complete the parallelogram $ABDC$, and prolong the diagonal till it intersects HG at F. If F is between H and G the pillar will be stable; if at H, it will be indifferent; if beyond H, it will be unstable. To find an expression for FG, draw DE perpendicular to AG. From the similar triangles ADE and AFG, we have,

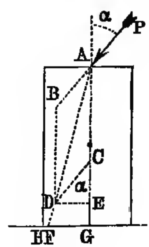

Fig. 52.

$$AE : AG :: DE : FG\,; \qquad \therefore\ FG = \frac{AG \times DE}{AE}.$$

But $AG = x$, $DE = P\sin a$, and $AE = W + P\cos a$, hence, we have,

$$FG = \frac{Px\sin a}{W + P\cos a};$$

And, since HG equals a, we have the following conditions for stability, indifference, and instability, respectively:

$$a > \frac{Px\sin a}{W + P\cos a}$$

$$a = \frac{Px\sin a}{W + P\cos a};$$

$$a < \frac{Px\sin a}{W + P\cos a}.$$

IV.—ELEMENTARY MACHINES.

Definitions and General Principles.

54. A machine is a contrivance by means of which a force applied at one point is made to produce an effect at some other point.

The applied force is called the **power**, and the force to be overcome the **resistance**; the source of the power is called the **motor.**

Some of the more common motors are *muscular effort,* as exhibited by man and beast in various kinds of work; the *weight and kinetic energy of water,* as shown in various kinds of water-mills; the *expansive force of vapors and gases,* as displayed in steam and caloric engines; the *force of air in motion,* as exhibited in the windmill, and in the propulsion of sailing vessels; the *force of magnetic attraction and repulsion,* as shown in the magnetic telegraph and other magnetic machines; the *elastic force of springs,* as shown in watches and various other machines. Of these the most important are *steam* and *water power.*

Applied and Useful Work.—Modulus.

55. Machines simply transmit and modify the action of forces. They add nothing to the work of the motor; on the contrary, they absorb and render inefficient much of the force which is impressed on them.

Of the *applied work,* a part is expended in overcoming *friction, stiffness of cords, bands* or *chains, resistance of the air,* and *adhesion of the parts.* This goes to *wear out* the machine. A second portion is expended in overcoming

shocks, arising from the nature of the work to be accomplished, as well as from imperfect connection of the parts, and from want of hardness and elasticity in the connecting pieces. This also goes to *strain and wear out* the machine, and to increase the waste already mentioned.

In any machine the quotient obtained by dividing the quantity of *useful*, or *effective work*, by the quantity of *applied work*, is called the **modulus** of the machine. As the resistances are diminished, the modulus increases, and the machine becomes more perfect.

Trains of Mechanism.

56. A machine usually consists of an assemblage of moving pieces called **elements**, kept in position by a connected system called a **frame**. Of the moving pieces, that which receives the power is called the **recipient** or **prime mover**, that which performs the work, is called the **operator** or **tool**, and the connecting pieces constitute what is called a **train of mechanism**. Of two consecutive elements, that which imparts motion is called a **driver**, and that which receives *motion* is called a **follower**. Each piece, except the extremes, is a *follower*, with respect to that which precedes, and a *driver*, with respect to that which follows.

In studying a train of mechanism we first find the relation between the power and resistance for each element neglecting hurtful resistances. We then modify these results so as to take account of all resistances, such as friction, adhesion, stiffness of cords, and atmospheric resistance. Having found the relation between the power and resistance for each piece, we begin at one extreme and combine them, recollecting, that the *resistance* with respect to each driver is equal to the *power* with respect to its follower.

We might also find the modulus of each element, and take

the product of these partial moduli as the modulus of the entire machine.

The Mechanical Powers.

57. The *elements* to which all machines can be reduced, are sometimes called **mechanical powers.** They are seven in number—viz., the **cord,** the **lever,** the **inclined plane,** the **pulley,** the **wheel and axle,** the **screw,** and the **wedge.** The first three are **simple elements**; the pulley, and the wheel and axle, are combinations of the cord and lever; the screw is a combination of two inclined planes twisted round an axle; and the wedge is a simple combination of two inclined planes.

In finding the relation between the power and resistance we shall, in the first instance, not only neglect all hurtful resistances, but we shall also suppose cords, levers, and connecting links, to be destitute of weight.

The Cord.

58. Let AB be a cord solicited by two forces, P and R, applied at its extremities, A and B. In order that the cord may be in equilibrium, it is evident, in the first place, that the forces must act in the direction of the cord, and in such manner as to stretch it, otherwise the cord would bend; and in the second place, the forces must be equal, otherwise the greater would prevail, and motion would ensue. Hence, if two forces applied at the extremities of a cord are in equilibrium, *the forces are equal and directly opposed.*

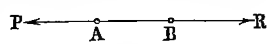

Fig. 53.

Let AB be a cord solicited by groups of forces applied at its extremities. In order that these forces may be in equilibrium, the resultants of the groups at A and B must be *equal and directly opposed.*

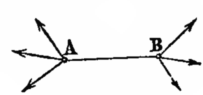

Fig. 54.

Let $ABCD$ be a cord, at the points A, B, C, D, of which groups of forces are applied. If these forces are in equilibrium through the intervention of the cord, there must necessarily be an equilibrium at each point, and this, whatever may be the lengths of AB, BC, and CD. If we make these infinitely small, the equilibrium will still subsist. But in that case the points A, B, C, and D, will coincide, and all the forces will be applied at a single point. Hence, *we conclude, that a system of forces applied in any manner at points of a cord will be in equilibrium, when, if applied at a single point without change of intensity or direction, they will maintain each other in equilibrium.*

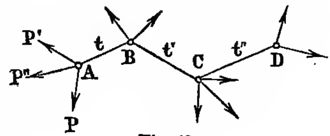

Fig. 18.

From what precedes, we see that the function of a cord in mechanism is simply to transmit forces, without modifying them in any manner.

The **tension** of a cord is the force by which two of its adjacent parts are urged to separate. If a cord be solicited in opposite directions by equal forces, its tension is measured by either force. If the forces are unequal, the tension is measured by the less.

The Lever.

59. A **lever** is an inflexible bar, free to turn around an axis, called the **fulcrum**.

Levers are sometimes divided into three classes, according to the relative positions of the power, resistance, and fulcrum.

In the **first class** the fulcrum is between the power and resistance, as in the ordinary balance; in the **second class** the resistance is between the power and the fulcrum, as in the ordinary nut cracker; in the **third class** the power is

between the resistance and the fulcrum. Strictly speaking, the last two are only varieties of a single class.

1st Class.

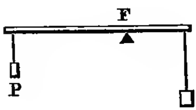

Fig. 56.

2d Class.

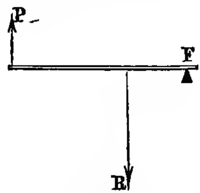

Fig. 57.

3d Class.

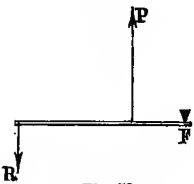

Fig. 58.

Levers may be *curved*, or *straight;* and the power and resistance may be either parallel or oblique to each other. We shall suppose the power and resistance to be perpendicular to the fulcrum ; otherwise we might conceive each to be resolved into two components, one perpendicular, and the other parallel, to this axis. The latter would tend to make the lever slide along the axis, developing hurtful resistance, while the former alone would tend to turn the lever about the fulcrum.

The perpendicular distances from the fulcrum to the lines of direction of the power and resistance, are called **lever arms.** In the bent lever MFN, the perpendicular distances FA, and FB, are the lever arms of P and R.

To determine the conditions of equilibrium of the lever, denote the power by P, the resistance by R, and their lever arms by p and r. We have the case of a body restrained by an axis, and if we take this as the axis of moments, we shall have for the condition of equilibrium (Art. 31),

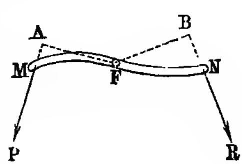

Fig. 59.

$$Pp = Rr; \quad \text{or,} \quad P : R :: r : p \; . \; . \; . \; . \; . \quad (45)$$

That is, *the power is to the resistance, as the lever arm of the resistance, is to the lever arm of the power.*

This relation holds good for every kind of lever.

If P and R intersect, their resultant will pass through the point of intersection, and also through F; its value may be found by the parallelogram of forces.

If several forces act upon a lever at different points, all being perpendicular to the direction of the fulcrum, they will be in equilibrium, when *the algebraic sum of their moments, with respect to the fulcrum, is equal to* 0. This principle enables us to take into account the *weight* of the lever, which is to be regarded as a vertical force through the centre of gravity of the lever.

MECHANICAL ADVANTAGE.—In any machine the quotient of the resistance by the power is called the **mechanical advantage.** Thus, in the case above considered, we have, for the mechanical advantage,

$$\frac{R}{P}=\frac{p}{r} \quad . \quad . \quad . \quad . \quad . \quad (46)$$

When $R > P$ there is said to be a *gain* of mechanical advantage, and when $R < P$ there is a *loss*. Because there is an equilibrium between R and P, their elementary quantities of work must be equal, from the theorem of work (Eq. 25); hence, if $R > P$, we have, $\delta r < \delta p$, and if $R < P$, we have, $\delta r > \delta p$. These principles apply to every case in which there is an equilibrium between the power and resistance.

Friction.

60. Friction is the resistance that one body experiences in moving on another when the two bodies are pressed together by a force that is normal to both.

This resistance arises from inequalities in the surfaces of the bodies, the projections of the one falling into the depressions of the other. To overcome the resistance, a sufficient force must be applied either to break off or bend down the

projecting points, or else to drag the body over them; this force must be equal and directly opposed to *force of friction,* which acts tangentially to the two surfaces.

Between certain bodies, friction is somewhat different when motion is just beginning, from what it is when motion has been established. The friction developed when a body is passing from a state of rest to a state of motion, is called **friction of quiescence**; that between bodies in motion, is called **friction of motion.**

The following **laws of friction** have been established by experiment, viz.:

First, *friction of quiescence between the same bodies is proportional to the normal pressure, and independent of the extent of the surfaces in contact.*

Secondly, *friction of motion between the same bodies, is proportional to the normal pressure, and independent, both of the extent of surface of contact, and of the velocity of the moving body.*

Thirdly, *for compressible bodies, friction of quiescence is greater than friction of motion: for bodies which are incompressible, the difference is scarcely appreciable.*

Friction may be diminished by the interposition of unguents, which fill up the cavities, and so diminish the roughness of the rubbing surfaces. For slow motions and great pressures, the more substantial unguents are used, such as lard, tallow, and certain mixtures; for rapid motions, and light pressures, oils are generally employed.

Methods of finding the Coefficient of Friction.

61. The quotient obtained by dividing the force of friction by the normal pressure, is called the **coefficient of friction**; its value for any two substances, may be determined as follows:

Let AB be a horizontal plane formed of one of the substances, and O a cubical block of the other. Attach a string, OC, to the block, so that its direction shall pass through the centre of gravity, and be parallel to AB; let the string pass over a fixed pulley, C, and let a weight, F, be attached to its extremity.

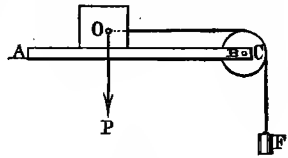

Fig. 60.

Increase F till O just begins to slide along the plane, then will F be the force of friction. Denote the normal pressure by P, and the coefficient of friction by f. From the definition, we have,

$$f = \frac{F}{P}. \quad . \quad . \quad . \quad (47)$$

In this manner, values for f may be found for different substances, and arranged in tables.

The value of f, for any substance, is its *coefficient* of friction. Hence, we may define the coefficient of friction to be *the friction due to a normal pressure of one pound.*

Having the normal pressure in pounds, and the coefficient of friction, the entire friction may be found by multiplying these quantities together.

There is a second method of finding the value of f, as follows:

Let AB be an inclined plane, formed of one of the substances, and O a block, of the other. Elevate the plane till the block just begins to slide down by its own weight. Denote the inclination, at this instant, by α, and the weight of O, by W. Resolve W into two components, one normal to the plane, and the other parallel to it. Denote the former by P, and the latter

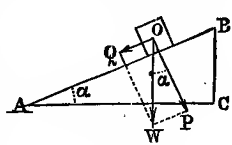

Fig. 61.

by Q. Since OW is perpendicular to AC, and OP to AB, the angle, WOP, is equal to α. Hence,

$$P = W \cos \alpha, \quad \text{and} \quad Q = W \sin \alpha.$$

The normal pressure being equal to $W \cos \alpha$, and the force of friction being $W \sin \alpha$, we shall have, from the principle already explained,

$$f = \frac{W \sin \alpha}{W \cos \alpha}, \quad \text{or,} \quad f = \tan \alpha \quad . \quad . \quad . \quad . \quad . \quad (48)$$

The angle α is called the **angle of friction.** The values of f for a few of the more common cases are given in the following

TABLE.

Bodies between which friction takes place.	*Coefficient of friction.*
Iron on oak	.62
Cast-iron on oak	.49
Oak on oak, fibres parallel	.48
Do., do., greased	.10
Cast-iron on cast-iron	.15
Wrought-iron on wrought-iron	.14
Brass on iron	.16
Brass on brass	.20
Wrought-iron on cast-iron	.19
Cast-iron on elm	.19
Soft limestone on the same	.64
Hard limestone on the same	.38

Equilibrium Bordering on Motion.

62. Let A be a movable body, resting on a fixed body B; let P be the resultant of all the forces acting on A, when it is on the eve of sliding along B in the direction from D to C, and let R, equal and directly opposed to P, be the corresponding reaction; that there may be no tendency to

rotation, the direction of P must intersect the surface, DC, at some point within the polygon of support. At A, the point of application of P, draw AN, normal to DC.

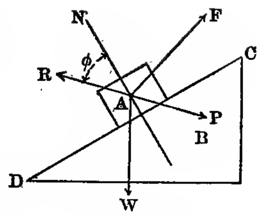

Fig. 62.

Now, let P be resolved into two components, one in the direction of the normal, and the other in the direction of the tangent DC; if we denote the angle between P and the normal by ϕ, the former will be equal to $P \cos \phi$, and the latter to $P \sin \phi$.

The normal component of P multiplied by f will be the measure of the entire friction between A and B, which is directed from C towards D; the tangential component of P exactly balances the force of friction and is directed from D towards C; hence, we have

$$P \cos \phi \times f = P \sin \phi, \text{ or } f = \tan \phi, \text{ or } \phi = \tan^{-1} f \quad . \quad . \quad . \quad (49)$$

That is, *when a body is on the eve of sliding along a second body, the reaction of the second body lies on the side of the normal opposite to the direction of incipient motion and makes an angle with it equal to the angle of friction.*

Method of Finding the Modulus.

63. The **modulus** of a machine is the quotient of *the elementary quantity of work of the resistance* by *the elementary quantity of work of the power.* Assuming the same notation as before and denoting the modulus by M, we have

$$M = \frac{R\delta r}{P\delta p} = \frac{R}{P} \times \frac{\delta r}{\delta p} \quad . \quad . \quad . \quad . \quad . \quad (50)$$

When friction is considered, P will be a function of f, that is, it will depend upon f for its value; let this value of P be

denoted by P_f; it is obvious that the values of R, δr, and δp will be entirely independent of f. When friction is not considered, P is obviously what P_f becomes when $f = 0$; denote this value of P by P_0. When P_0 and R are in equilibrium, we have

$$P_0 \delta p = R \delta r, \quad \text{or} \quad \frac{\delta r}{\delta p} = \frac{P_0}{R}.$$

Substituting in (50), we have

$$M = \frac{R}{P_f} \times \frac{P_0}{R} = \frac{P_0}{P_f} \quad . \quad . \quad . \quad . \quad . \quad (51)$$

Hence, we have the following rule for finding the modulus of a machine when friction is taken into account:

Find the value of P in terms of R and f; in this result make f equal to 0; then divide the latter result by the former.

To Find the Modulus of a Lever.

64. Let AB represent a lever and suppose its fulcrum to be a solid cylinder turning in a cylindrical box; also, suppose both lever and fulcrum to be horizontal. Denote the power by P, the resistance by R, the weight of the lever by W, and let the distances of these forces from the centre of the fulcrum be p, r, and w. If P, R, and W are vertical the resultant reaction OQ will pass through the point of contact O and will be directed vertically upward.

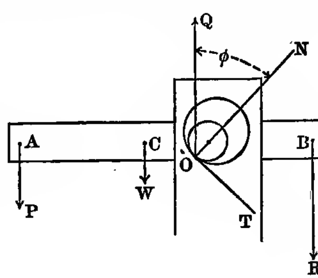

Fig. 63.

When P is on the eve of overcoming all the resistances, it will be equal to P_f of the preceding article, the fulcrum

will be on the eve of sliding in the direction OT, and the normal ON, which passes through the axis of the fulcrum, must be so situated as to make the angle NOQ, denoted by ϕ, equal to the angle of friction (Art. 62).

Take O as a centre of moments and denote the radius of the fulcrum by ρ; the corresponding lever arm of P_f will be $p - \rho \sin \phi$, that of R will be $r + \rho \sin \phi$, and that of W will be $w - \rho \sin \phi$; and because the forces are in equilibrium, we shall have, from Art. 31,

$$P_f (p - \rho \sin \phi) + W (w - \rho \sin \phi) = R (r + \rho \sin \phi),$$

from which equation we find,

$$P_f = \frac{R (r + \rho \sin \phi) - W' (w - \rho \sin \phi)}{p - \rho \sin \phi} \quad . \quad . \quad . \quad . \quad (52)$$

Making $f = 0$, which makes $\sin \phi = 0$, we have

$$P_0 = \frac{Rr - Ww}{p} \quad . \quad . \quad . \quad . \quad . \quad (53)$$

Substituting the values of P_0 and P_f in equation 51, we have

$$M = \frac{Rr - Ww}{p} \times \frac{p - \rho \sin \phi}{R (r + \rho \sin \phi) - W (w - \rho \sin \phi)} \quad . \quad . \quad . \quad (54)$$

NOTE.—We have $\sin \phi = \dfrac{f}{\sqrt{1 + f^2}}$ from Trigonometry.

EXAMPLE.—Let $R = 100$ *lbs.*, $W = 5$ *lbs.*, $r = 20$ *in.*, $p = 50$ *in.*, $w = 15$ *in.*, $\rho = 2$ *in.*, and $\sin \phi = 3$ ($f = .314$ nearly): What is the modulus? *Ans.* .96 *nearly*, that is, about 96% of the applied work becomes effective.

NOTE.—In many cases the value of P_f may be determined by experiment and the value of P_0 may be found from the equation of equilibrium when friction is neglected.

The Compound Lever.

65. **A compound lever** is a combination of simple levers AB, BC, CD, so arranged that the resistance in one acts as a power in the next, throughout the combination. Thus, a power P produces at B a resistance R', which, in turn, produces at C a resistance R'', and so on. Let us assume the notation of the figure. From the principle of the simple lever, we have the relations,

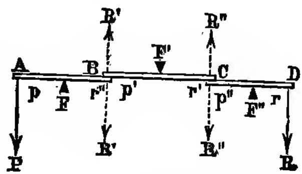

Fig. 64.

$$Pp = R'r'', \; R'p' = R''r', \; R''p'' = Rr.$$

Multiplying these equations, member by member, and striking out common factors, we have

$$Ppp'p'' = Rrr'r''; \text{ or, } P : R :: rr'r'' : pp'p'' \; . \; . \; . \; . \; . \; (55)$$

And similarly for any number of levers.

Hence, in the compound lever, *the power is to the resistance as the continued product of the alternate arms of lever, commencing at the resistance, is to the continued product of the alternate arms of lever, commencing at the power.*

Modulus.—If p, p', p'', are each equal to 10; if r, r, r'' are each equal to 2; and if R equals 100 *lbs.*, we have $P_0 = .8 lb.$ If we find by experiment that a force of 1 *lb.* applied at A will just overcome a resistance of 100 *lbs.* applied at D, we have $P_1 = 1$ *lb.* Then, to find the modulus, substitute these in (51), and we have

$$M = \frac{.8}{1} = 80\% \; . \; . \; . \; . \; . \; (56)$$

that is, 80% of the applied force becomes effective.

The Elbow-Joint Press.

66. Let BD and BC represent equal bars, having hinge-points at B, C, and D, and let DE be a bar that works back and forth through a guide between E and D. Let P be the power applied at B, and perpendicular to DC. When P acts to depress B, the link BC turns about C, and the force transmitted through the link BD causes the piece DE to move towards F, so as to compress a body placed between E and F. This machine is called the **elbow-joint press,** and is used in some kinds of printing, in moulding bullets, in stamping coins, in crushing stone, and in many other like kinds of work.

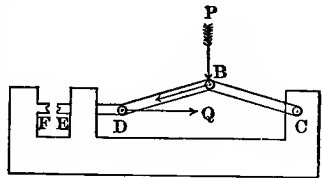

Fig. 65.

To find the mechanical advantage, let us resolve P into two components in the direction of the links BC and BD; these components are equal, and if we denote one of them by Q, and the angle, DBC, by β, we have, from (14),

$$P^2 = Q^2 + Q^2 + 2Q^2 \cos\beta;$$

whence
$$Q = \frac{P}{\sqrt{2(1+\cos\beta)}}$$

If we denote the angle QDB by α, the effective component of Q is $Q\cos\alpha$, and this is equal and directly opposed to the resistance R; hence,

$$\frac{R}{P} = \frac{\cos\alpha}{\sqrt{2(1+\cos\beta)}} \quad \ldots\ldots \quad (57)$$

As B approaches the line DC, $\cos\alpha$ approaches 1, and $\cos\beta$ approaches -1; at the limit $\cos\alpha = 1$, and $\cos\beta = -1$,

and the mechanical advantage becomes *infinite.* It is to be noted that the elementary space through which the pressure is exerted varies inversely as the mechanical advantage; hence, at the limit it is 0.

To find the modulus of the reciprocating piece DE: assume the figure in which AB represents the part within the guide, O being its centre. In addition to the notation already adopted, let the length of AB be denoted by $2l$, its breadth by $2b$, and the distance, OD, by d.

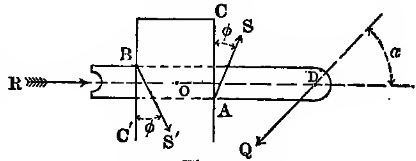

Fig. 66.

The tendency of Q is to produce rotation, pressing the piece downward at A, and upward at B; when Q is on the eve of overcoming R, there will be a reaction S at A, and a reaction S' at B, each of which, from Art. 62, will lie on that side of the corresponding normal which is opposite to the direction of incipient motion, and in each case the inclination to the normal will be equal to ϕ, the angle of friction.

At the instant in question there will be an equilibrium between Q, S, S', and R; consequently, the algebraic sum of their components in the direction of OD, and in the direction perpendicular to OD, will be separately equal to 0, and also the algebraic sum of their moments with respect to O will be equal to 0. Hence, we have,

$$Q \cos \alpha = R + (S + S') \sin \phi \quad . \quad . \quad . \quad . \quad . \quad (a)$$

$$Q \sin \alpha = (S - S') \cos \phi \quad . \quad . \quad . \quad . \quad . \quad (b)$$

$$Q \sin \alpha \times d - (S + S') \cos \phi \times l - (S - S') \sin \phi \times b = 0 \quad . \quad . \quad . \quad . \quad . \quad (c)$$

Finding $S + S'$ and $S - S'$ fron (a) and (b), and substituting in (c), we have,

$$Q \sin \alpha \times d - \frac{Q \cos \alpha - R}{\sin \phi} \cos \phi \times l - \frac{Q \sin \alpha}{\cos \phi} \sin \phi \times b = 0.$$

Making $\frac{\sin \phi}{\cos \phi} = f$, factoring, and clearing, we have,

$$Q (fd \sin \alpha - bf^2 \sin \alpha - l \cos \alpha) + Rl = 0.$$

Solving with respect to Q, which is the same as Q_f, we have,

$$Q_f = \frac{Rl}{l \cos \alpha - fd \sin \alpha + bf^2 \sin \alpha} \quad . \; . \; . \; . \; . \quad (d)$$

Making $f = 0$, we have,

$$Q_0 = \frac{Rl}{l \cos \alpha} = \frac{R}{\cos \alpha} \quad . \; . \; . \; . \; . \quad (f)$$

Hence, from Eq. (51), after reduction, we have,

$$M = \frac{l - fd \tan \alpha + bf^2 \tan \alpha}{l} \quad . \; . \; . \; . \; . \quad (58)$$

Hence, the modulus varies with α, as was to have been expected; at the limit, $M = 1$.

Weighing Machines.

67. Nearly all **weighing machines** depend on the principle of the lever; the resistance is the weight to be determined, and the power is a counterpoising weight of known value.

There are two principal classes of weighing machines: in the *first*, the lever arm of the power is constant, and the power varies; in the *second*, the power is constant, and its lever arm varies. The ordinary balance is an example of the first class, and the steelyard of the second.

The Common Balance.

68. The **common balance** consists of a lever, AB, called

the beam, having a knife-edge fulcrum, F, and two **scale-pans**, D and E, suspended from its extremities by means of knife-edge joints at A and B. The beam is supported by a standard, FK, resting on a foot-plate, L. The standard is made vertical by leveling screws passing through the foot-plate. The knife-edges and their supports are of hardened steel; and to prevent unnecessary wear, an arrangement is made for throwing them from their bearings when not in use. A needle, N, playing in front of a graduated scale, GH, shows the amount of deflection of the beam.

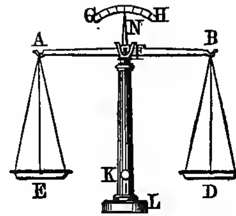

Fig. 67.

A good balance should fulfill the following conditions: 1°, it should be **true**; 2°, it should be **stable**—that is, when the beam is deflected it should tend to return to a horizontal position; 3°, it should be **sensitive**—that is, it should be deflected from the horizontal by a small force.

In order that a balance may be *true*, its lever arms must be equal in length, and when the beam is horizontal, both the beam and scale-pans must be symmetrical with respect to two planes through the centre of gravity of the beam, the first plane being perpendicular to the beam, and the second perpendicular to the fulcrum.

In order that it may be *stable*, the centre of gravity of the beam must be below the fulcrum, and the line joining the points of suspension of the scale-pans must not pass above the fulcrum.

In order that it may be *sensitive*, the line joining the points of suspension must not pass below the fulcrum, the lever arms must be as long, and the beam as light as is consistent with strength and stiffness, the knife-edges must be horizontal and parallel to each other, and the friction at the joints

must be as small as possible. The sensitiveness of a balance diminishes as the load increases.

The true weight of a body may be found by a balance whose lever arms are not equal, by means of the principle demonstrated below.

Denote the length of the lever arms, by r and r', and the weight of the body, by W. When the weight W is applied at the extremity of the arm r, denote the counterpoising weight by W'; and when it is at the extremity of the arm r', denote the counterpoising weight by W''. We shall have, from the principle of the lever,

$$Wr = W'r', \text{ and } Wr' = W''r.$$

Multiplying these equations, member by member, we have,

$$W^2rr' = W''W'rr'; \quad \therefore \quad W = \sqrt{W'\,W''}; \qquad (59)$$

that is, *the true weight is equal to the square root of the product of the apparent weights.*

A still better method, and one that is more free from the effect of errors in construction, is to place the body to be weighed in one scale, and put weights in the other, till the beam is horizontal; then remove the body to be weighed, and replace it by known weights, till the beam is again horizontal; the sum of the replacing weights will be the weight required. If, in changing the load, the positions of the knife-edges be not changed, this method is almost perfect.

The Steelyard.

69. The **steelyard** is an instrument for weighing bodies. It consists of a lever, AB, called the beam; a fulcrum, F; a scale-pan, D, attached at the extremity of one arm; and a known weight, E, movable along the other arm. We shall suppose the weight of E to be 1 lb. This instrument is sometimes more convenient than the balance, but it is not so accu-

rate. The conditions of sensitiveness are essentially the same as for the balance.

To graduate the instrument, place a pound weight in the pan, *D*, and move the counterpoise *E* till the beam rests horizontal—let that point be marked 1; next place a 10 lb. weight in the pan, and move the counterpoise *E* till the beam is again horizontal, and let that point be marked 10; divide the intermediate space into nine equal parts, and mark the points of division as shown in the figure. These spaces may be subdivided at pleasure, and the scale extended to any desirable limit. We have supposed the centre of gravity to coincide with the fulcrum; when this is not the case, the weight of the instrument must be taken into account as a force applied at its centre of gravity. We may then graduate the beam by experiment, or we may compute the lever arms, corresponding to different weights, by the principle of moments.

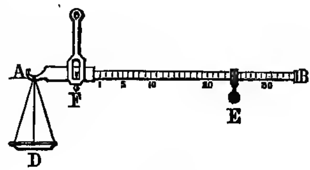

Fig. 68.

To weigh a body with the steelyard, place it in the scale-pan, and move the counterpoise *E* along the beam till an equilibrium is established; the mark on the beam will indicate the weight.

The Compound Balance.

70. Compound balances are used in weighing heavy articles, as merchandise, coal, freight for shipping, and the like. A great variety of combinations have been employed, one of which is shown in the figure.

AB is a platform on which is placed the object to be weighed; *BC* is a guard firmly attached to the platform; the platform

is supported on the knife-edge fulcrum E, and the piece D, through the medium of a brace CD; GF is a lever turning about the fulcrum F, and suspended by a link from the point L; LN is a lever having its fulcrum at M, and sustaining

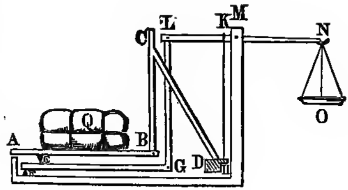

Fig. 69.

the piece D by a link KH; O is a scale-pan suspended from the end N of the lever LN. The instrument is so constructed, that

$$EF : GF :: KM : LM;$$

and we shall suppose that KM is made equal to $\frac{1}{10}$ of MN. The parts are so arranged that the beam LN shall rest horizontally when no weight is placed on the platform.

If, now, a body Q be placed on the platform, a part of its weight will be thrown on the piece D, and, acting downward, will produce an equal pressure at K. The remaining part will be thrown on E, and, acting on the lever FG, will produce a downward pressure at G, which will be transmitted to L; but, on account of the relation given by the above proportion, the effect of this pressure on the lever LN will be the same as though the pressure thrown on E had been applied directly at K. The final effect is, therefore, the same as though the weight of Q had been applied at K, and, to counterbalance it, a weight equal to $\frac{1}{10}$ of Q must be placed in the scale-pan O.

To weigh a body, place it on the platform, and add weights to the scale-pan till LN is horizontal, then 10 times the sum of the weights added will be the weight required. By applying the principle of the steelyard to this balance, objects may be weighed by using a constant counterpoise.

EXAMPLES.

1. In a lever of the first class, the lever arm of the resistance is $2\frac{2}{3}$ inches, that of the power, $33\frac{1}{3}$, and the resistance 100 *lbs.* What power is necessary to hold the resistance in equilibrium? *Ans.* 8 *lbs.*

2. Four weights of 1, 3, 5, and 7 *lbs.*, are suspended from points of a straight lever, eight inches apart. How far from the point of application of the first weight must the fulcrum be situated, that the weights may be in equilibrium?

SOLUTION.—Let x denote the required distance. Then, from Art. (31),

$$1 \times x + 3(x - 8) + 5(x - 16) + 7(x - 24) = 0;$$

$$\therefore \quad x = 17 \text{ in. } \textit{Ans.}$$

3. A lever, of uniform thickness, and 12 feet long, is kept horizontal by a weight of 100 *lbs.* applied at one extremity, and a force P applied at the other extremity, so as to make an angle of 30° with the horizon. The fulcrum is 20 inches from the point of application of the weight, and the weight of the lever is 10 *lbs.* What is the value of P, and what is the pressure on the fulcrum?

SOLUTION.—The lever arm of P is equal to 124 in. $\times \sin 30° = 62$ *in.*, and the lever arm of the weight of the lever is 52 *in.* Hence,

$$20 \times 100 = 10 \times 52 + P \times 62. \qquad \therefore \quad P = 24 \text{ lbs. nearly.}$$

We have, also,

$$R = \sqrt{X^2 + Y^2} = \sqrt{(110 + 24 \sin 30°)^2 + (24 \cos 30°)^2}.$$

$$\therefore \quad R = 123.8 \text{ lbs.};$$

and, $$\cos a = \frac{X}{R} = \frac{20.785}{123.8} = .16789;$$

$$\therefore \quad a = 80° \ 20' \ 02''.$$

4. A heavy lever rests on a fulcrum 2 feet from one end, 8 feet from the other, and is kept horizontal by a weight of 100 *lbs.*, applied at the first end, and a weight of 18 *lbs.*, applied at the other end. What is the weight of the lever, supposed of uniform thickness throughout?

SOLUTION.—Denote the required weight by x; its arm of lever is 3 *ft.* We have, from the principle of the lever,

$$100 \times 2 = x \times 3 + 18 \times 8; \qquad \therefore \quad x = 18\tfrac{2}{3} \text{ lbs. } \textit{Ans.}$$

5. Two weights keep a horizontal lever at rest; the pressure on the fulcrum is 10 *lbs.*, the difference of the weights is 4 *lbs.*, and the difference of lever arms is 9 inches. What are the weights, and their lever arms?

Ans. The weights are 7 *lbs.* and 3 *lbs.*; their lever arms are $15\frac{3}{4}$ *in.*, and $6\frac{3}{4}$ *in.*

6. The apparent weight of a body weighed in one pan of a false balance is $5\frac{1}{2}$ *lbs.*, and in the other pan it is $6\frac{6}{11}$ *lbs.* What is the true weight?

$$W = \sqrt{\tfrac{11}{2} \times \tfrac{72}{11}} = 6 \text{ lbs.} \quad \textit{Ans.}$$

The Inclined Plane.

71. An **inclined plane** is a plane that is inclined to the horizon.

In this machine the *resistance* is the weight of a body acting vertically downward, and the *power* is a force applied to the body, either to prevent motion down the plane, or to produce motion up the plane. The power may be applied in any direction, but we shall suppose it to be in the vertical plane that is perpendicular to the inclined plane.

To find the mechanical advantage when friction is not considered. Let BA be the plane, O a body resting on it whose weight is denoted by R, and let P be the force necessary to hold it in equilibrium. The resultant of P and R, denoted by Q, must pass through the polygon of support and be perpendicular to AB (Art. 53). If we denote the angle between P and R by β, and the angle between Q and R, which is equal to the inclination of the plane, by α, the angle between P and Q will be equal to $\beta - \alpha$, and from equation (15) we shall have

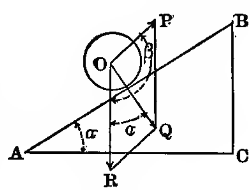

Fig. 70.

$$P : R :: \sin \alpha : \sin (\beta - \alpha),$$

or, $$\frac{R}{P} = \frac{\sin(\beta - \alpha)}{\sin \alpha} \quad \ldots \ldots \quad (60)$$

If the power is parallel to the plane $\beta - \alpha$, or ϕ, will be equal to 90°, or sin $(\beta - \alpha)$ will be equal to 1, and because $\sin \alpha = \frac{CB}{AB}$ equation (60) becomes

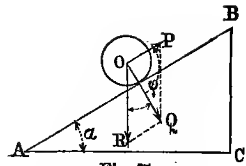
Fig. 71.

$$\frac{R}{P} = \frac{AB}{CB}, \quad \ldots \quad (61)$$

That is, *the power is to the resistance as the height of the plane is to its length.*

If the power is parallel to the base of the plane, that is, to the horizon, $\beta - \alpha$ will be equal to $90° - \alpha$, and because $\cot \alpha = \frac{AC}{CB}$ equation (60) will become

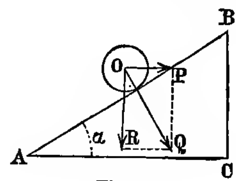
Fig. 72.

$$\frac{R}{P} = \frac{AC}{CB} \quad \ldots \ldots \quad (62)$$

That is, *the power is to the resistance as the height of the plane is to its base.*

If α increase, the value of P will increase, and when α becomes 90°, P becomes infinite; that is, no finite horizontal force can sustain a body against a vertical wall, without the aid of friction.

Examples.

1. A power of 1 *lb.*, acting parallel to an inclined plane, supports a weight of 2 *lbs.* What is the inclination of the plane? *Ans.* 30°.

2. The power, resistance, and normal pressure, in the case of an inclined plane, are, respectively, 9, 13, and 6 *lbs.* What is the inclination of the plane, and what angle does the power make with the plane?

Ans. Inclination of power to plane $= \phi - 90° - a = 28° 46' 54''$.

SOLUTION.—If we denote the angle between the power and resistance by ϕ, and the inclination of the plane by α, we have, from Eq. (12),

$$6 = \sqrt{13^2 + 9^2 + 2 \times 9 \times 13 \cos \phi};$$

$$\therefore \phi = 156° \, 8' \, 20''.$$

Also, from Eq. 12, for the inclination of the plane,

$$6 : 9 :: \sin 156° \, 8' \, 20'' : \sin \alpha ; \qquad \therefore \alpha = 37° \, 21' \, 26''.$$

3. A body is supported on an inclined plane by a force of 10 *lbs.*, acting parallel to the plane; but it requires a force of 12 *lbs.* to support it when the force acts parallel to the base. What is the weight of the body, and the inclination of the plane?

Ans. The weight is 18.09 *lbs.*, and the inclination 33° 33′ 25″.

Modulus of the Inclined Plane.

72. If a block O is placed upon the inclined plane AB whose *inclination*, α, is greater than the *angle of friction*, it will slide down unless supported by some force; if the *inclination* of the plane is less than to the *angle of friction* it will have no tendency to move either up or down the plane unless acted upon by some other force than its weight.

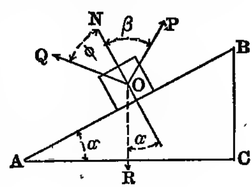

Fig. 73.

To find the modulus when friction is considered: let us suppose the body to be on the eve of motion up the plane under the action of the force P, which makes an angle with the normal, ON, equal to β. The reaction, Q, due to the action of P and R will, from Art. 62, be directed as shown in the figure, ϕ being equal to the angle of friction.

Because there is an equilibrium between P, R, and Q, we shall have, from Eq. (16),

$$P : R :: \sin [180° - (\phi + \alpha)] : \sin (\phi + \beta);$$

or, $$P : R :: \sin \phi \cos \alpha + \sin \alpha \cos \phi : \sin \phi \cos \beta + \sin \beta \cos \phi \quad . \; . \; . \; . \; . \quad (63)$$

Dividing both terms of the second couplet by $\cos\phi$, remembering that P is now equal to P_f, and replacing $\tan\phi$ by f, we have

$$P_f = R\frac{f\cos\alpha + \sin\alpha}{f\cos\beta + \sin\beta} \quad . \quad . \quad . \quad . \quad . \quad (64)$$

Making $f = 0$, we have

$$P_0 = R\frac{\sin\alpha}{\sin\beta} \quad . \quad . \quad . \quad . \quad . \quad (65)$$

Substituting in (51), we have for the modulus,

$$M = \frac{\sin\alpha}{\sin\beta} \cdot \frac{f\cos\beta + \sin\beta}{f\cos\alpha + \sin\alpha} = \frac{f\cot\beta + 1}{f\cot\alpha + 1} \quad . \quad . \quad . \quad . \quad . \quad (66)$$

The **line of least traction** is the line along which P_f is the least possible; in this case the denominator of the value of P_f must be a maximum: assuming

$$u = f\cos\beta + \sin\beta$$

and differentiating with respect to β, we have

$$\frac{du}{d\beta} = -f\sin\beta + \cos\beta = 0; \quad \therefore \beta = \cot^{-1}f.$$

Differentiating again

$$\left(\frac{d^2u}{d\beta^2}\right)_f = (-f\cos\beta - \sin\beta)_f < 0.$$

Hence, u is a maximum, and consequently P_f is a minimum, when the complement of β is equal to the angle of friction, that is, when the direction of P_f makes an angle with plane equal to the angle of friction.

Let the student find the modulus when P_f is on the eve of moving the body down the plane.

Work when the Traction is Parallel to the Plane.

73. If we make $\beta = 90°$ in Eq. 64, the *force of traction*

P_f will be parallel to the plane and will be directed from A towards B. In this case, we have

$$P_f = Rf \cos \alpha + R \sin \alpha.$$

Multiplying both members by AB, and replacing $AB \cos \alpha$ by AC, and $AB \sin \alpha$ by CB, it becomes

$$P_f \times AB = Rf \times AC + R \times CB \quad . \quad . \quad . \quad . \quad (67)$$

The first member is the work of the force of traction in drawing the body from A to B; the first term of the second member is the work that would be required to draw the body horizontally from A to C; and the second term is the work that would be required to lift the body vertically from C to P. Formula (67) finds an application in computing the work of a locomotive in drawing a train up an inclined plane.

The Pulley.

74. A **pulley** is a wheel having a groove around its circumference to receive a cord; the wheel turns on an axis at right angles to its plane, and this axis is supported by a frame called a **block**. The pulley is said to be *fixed*, when the block is fixed, and *movable*, when the block is movable. Pulleys are used singly, or in combinations.

Single Fixed Pulley.

75. In this machine the block is fixed. Denote the power by P, the resistance by R, and the radius of the pulley by r. It is plain that both the power and resistance should be at right angles to the axis. Hence, if we take the axis of the pulley as an axis of moments, we have (Art. 31), in case of equilibrium,

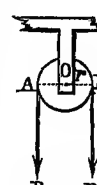

Fig. 74.

$$Pr = Rr; \text{ or, } P = R \quad . \quad . \quad . \quad . \quad (68)$$

Stiffness of Cordage.

76. Let O be a pulley, with a cord, AB, wrapped round its circumference; and suppose a force, P, applied at B, to overcome a resistance, R. As the rope winds on the pulley, its rigidity acts to increase the arm of lever of R, and to overcome this rigidity an additional force is required. This additional force may be represented by the expression,

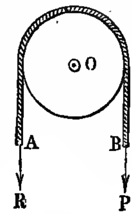

Fig. 75.

$$d\left(\frac{a + bR}{D}\right);$$

in which d depends on the character and size of the rope, a on its natural rigidity, bR on the rigidity due to the load, and D is the diameter of the wheel. The values of d, a, and b have been found by experiment for different kinds of rope, and tabulated.

The moment of the resistance is found by multiplying the preceding expression by its lever arm, $\frac{1}{2}D$, or r; denoting this moment by S, it may be written,

$$S = m + nR \quad . \quad . \quad . \quad . \quad . \quad (69)$$

Modulus of the Single Fixed Pulley.

77. Let the solid axle of the pulley turn in a cylindrical box as shown in the figure; also, suppose the axis to be horizontal. Let the power, P, supposed vertical, be applied at one extremity of a cord passing over the pulley AB, and the resistance R at the other extremity; denote the radius of the axle by ρ, the radius of the pulley by r, and the weight of the

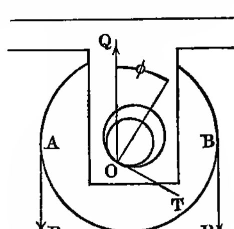

Fig. 75*.

pulley and cord (supposed to pass through the centre of the pulley) by W.

When P acts to turn the pulley the axle *rolls* up in its box until it reaches a point O, beyond which, if it be further displaced, it will *slide* back. At this point, when P is on the eve of overcoming the resistances, the reaction, Q, will be vertical and will make with the normal at O an angle equal to ϕ, the angle of friction, and we shall have, as in Art. 64,

$$P(r - \rho \sin \phi) = R(r + \rho \sin \phi) + W\rho \sin \phi \quad . \quad . \quad . \quad . \quad (70)$$

Solving with respect to P (now P_f), we have

$$P_f = R \frac{r + \rho \sin \phi + \frac{W}{R} \rho \sin \phi}{r - \rho \sin \phi} \quad . \quad . \quad . \quad . \quad . \quad (71)$$

Making $f = 0$, or $\sin \phi = 0$, we have

$$P_0 = R \quad . \quad . \quad . \quad . \quad . \quad (72)$$

Hence, we have

$$M = \frac{r - \rho \sin \phi}{r + \rho \sin \phi + \frac{W}{R} \rho \sin \phi} \quad . \quad . \quad . \quad . \quad . \quad (73)$$

In which $\sin \phi = \dfrac{f}{\sqrt{1 + f^2}}$.

If we would take stiffness of cordage into account we must add S, (Eq. 69), to the second member of (70), and when we make $f = 0$, we must also make $S = 0$, which gives for the modulus,

$$M = \frac{r - \rho \sin \phi}{r + \rho \sin \phi + \frac{1}{R}(W\rho \sin \phi + S)} \quad . \quad . \quad . \quad . \quad . \quad (74)$$

To find M *experimentally* we apply a weight P that is just on the eve of overcoming the resistances: then will the quotient of R by this weight be the value of the modulus.

Friction of Rope on a Fixed Axle.

78. Let the circle O represent the cross-section of a fixed cylindrical axle, and let $PLKR$ be a rope tangent to it at L and K. Suppose that a force P is on the eve of overcoming a resistance R applied as shown in the figure: there will be a tendency to motion all along the rope.

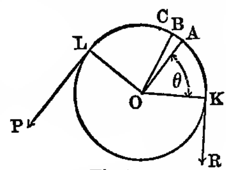

Fig. 76.

Let A, B, and C, be three consecutive points of the arc KL, and denote the angle KOA by θ: then will the angle between AB and the prolongation of CB be equal to $d\theta$. If we denote the tension on BA which acts to resist motion by t, the tension on BC tending to produce motion will be denoted by $t + dt$, and the reaction Q due to these forces will be situated as shown in Fig. 77, the angle ϕ being the angle of friction.

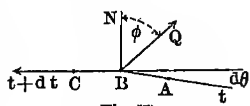

Fig. 77.

Because there is an equilibrium between t, $t + dt$, and Q, we shall have (Eq. 16),

$$t : t + dt :: \sin(90^\circ + \phi) : \sin(90^\circ - \phi + d\theta). \quad . \quad . \quad . \quad . \quad (75)$$

The third term of this proportion equals $\cos\phi$; the fourth term equals $\cos(\phi - d\theta)$, or to $\cos\phi \cos d\theta + \sin\phi \sin d\theta$; but $\cos d\theta = 1$ and $\sin d\theta = d\theta$; hence (75) may be written

$$t : t + dt :: \cos\phi : \cos\phi + \sin\phi \, d\theta; \quad . \quad . \quad . \quad . \quad . \quad (76)$$

or, by division,

$$t : dt :: \cos\phi : \sin\phi \, d\theta. \quad . \quad . \quad . \quad . \quad . \quad (77)$$

From (77) we have, after reduction,

$$\frac{dt}{t} = f\,d\theta. \quad . \quad . \quad . \quad . \quad . \quad (78)$$

Integrating the first member from $t = R$ to $t = P$, and the second member from $\theta = 0$ to $\theta = KOL$ (denoted by θ'), we have,

$$\int_R^P \frac{dt}{t} = \int_0^{\theta'} f\,d\theta, \quad \text{or} \quad l\frac{P}{R} = f\theta'. \quad . \quad . \quad . \quad . \quad . \quad (79)$$

Passing to numbers, (79) becomes

$$\frac{P}{R} = e^{f\theta'}, \quad \text{or} \quad P = Re^{f\theta'}. \quad . \quad . \quad . \quad . \quad . \quad (80)$$

In (80), $e = 2.71828$, f is the coefficient of friction between the rope and the cylinder, and θ' is the arc that measures the angle KOL. The unit of θ' is the radian which is very nearly equal to $57°.3$.

Example.—Let the rope be wrapped entirely around the cylinder, that is, let $\theta' = 2\pi$, and suppose that $f = .4$. What is the value of P?

Ans. $P = R\,(2.71828)^{2.513} = R \times 12.44$ *nearly*.

Single Movable Pulley.

79. In this pulley the block is movable. The resistance is applied by means of a hook attached to the block; one end of a rope, enveloping the lower part of the pulley, is attached at a fixed point, C, and the power is applied at its other extremity. We shall suppose, in the first place, that the two branches of the rope are parallel. Assuming the notation of Art. 75, neglecting friction, and taking A as a centre of moments, we have, in case of equilibrium,

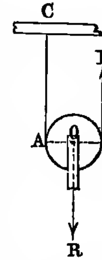

Fig. 78.

$$P \times 2r = Rr; \qquad \therefore \quad P = \tfrac{1}{2}R.$$

That is, *when the power and resistance are parallel, the power is one half the resistance.* The tension of the cord CA is the same as that of BP. It is, therefore, equal to one half the resistance. If the resistance of the point C be replaced by a force equal to P, the equilibrium will be undisturbed.

In the second place, let the two branches of the enveloping cord be oblique to each other. Suppose the resistance C to be replaced by a force equal to P, and denote the angle between the two branches of the rope by 2ϕ. If there is an equilibrium between P, P, and R, the horizontal components of P, P, will balance each other,

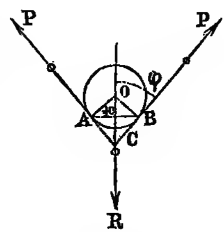

Fig. 79.

and the sum of their vertical components will be equal to R; hence, we have,

$$2P \cos \phi = R.$$

Draw the chord AB, and denote its length by c; draw, also, the radius OB. Then, because OR is perpendicular to AB and BP to OB, the angle ABO is one half ACB, or equal to ϕ. Hence,

$$\cos \phi = \tfrac{1}{2}c \div r.$$

Substituting in the preceding equation, and reducing, we have,

$$\frac{R}{P} = \frac{c}{r};$$

$$\therefore \quad P : R :: r : c. \quad . \quad . \quad . \quad . \quad . \quad (81)$$

That is, *the power is to the resistance, as the radius of*

the pulley, is to the chord of the arc enveloped by the rope.

When the chord of the enveloped arc is greater than the radius, there is a gain of *mechanical advantage;* when less, there is a loss.

If the chord is equal to the diameter, we have, as before,

$$P = \tfrac{1}{2}R.$$

Combination of Movable Pulleys.

80. The figure represents a combination of movable pulleys, in which there are as many cords as pulleys; one end of each cord is attached at a fixed point, the other end being fastened to the hook of the next pulley in order, up to the last cord, at the second extremity of which the power is applied.

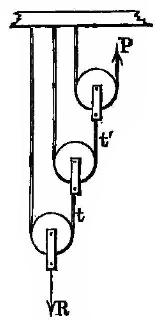

Fig. 80.

Denote the tension of the cord between the first and second pulley by t, that of the cord between the second and third pulley by t'. From the preceding Article, we have,

$$t = \tfrac{1}{2}R; \quad t' = \tfrac{1}{2}t; \quad P = \tfrac{1}{2}t'.$$

Multiplying these equations together, member by member, and reducing, we have,

$$P = (\tfrac{1}{2})^3 R.$$

Had there been n pulleys in the combination, we should have obtained, in a similar manner,

$$P = (\tfrac{1}{2})^n \cdot R;$$

$$\therefore P : R :: 1 : 2^n. \quad . \quad . \quad . \quad . \quad . \quad (82)$$

That is, *the power is to the resistance, as* 1 *is to* 2^n, n *denoting the number of pulleys.*

Combinations of Pulleys in Blocks.

81. These combinations are effected in various ways. In most cases, but one rope is employed, which, being attached to a hook of one block, passes round a pulley in the other block, then round one in the first, and so on, from block to block, till it has passed round each pulley in the system. The power is applied at the free end of the rope. Sometimes the pulleys in each block are placed side by side, sometimes one above another, as in the figure, in which case the inner ones are made smaller than the outer ones. The conditions of equilibrium are the same in both cases. To deduce the conditions of equilibrium in the case represented, denote the power by P, and the resistance by R, friction being neglected. When there is an equilibrium, the tension of each branch of the rope that aids in supporting the resistance must be equal to P; but, since the last pulley simply serves to change the direction of the force P, there will be four such branches in the case considered; hence, we shall have,

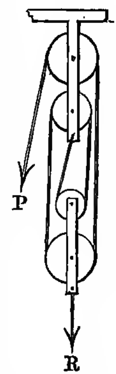

Fig. 81.

$$4P = R, \quad \text{or,} \quad P = \tfrac{1}{4}R.$$

Had there been n pulleys in the combination, there would have been n supporting branches, and we should have had,

$$nP = R, \quad \text{or,} \quad P : R :: 1 : n. \quad . \quad . \quad . \quad . \quad . \quad (83)$$

That is, *the power is to the resistance, as* 1 *is to the number of branches of the rope that support the resistance.*

In equation (83), P (which we may call the *theoretical*

power) is the same as P_0 in Eq. 51. The value of P_f in the same equation is the value of P which is found by experiment to be just capable of overcoming all the resistances.

EXAMPLE.—In the combination shown in Fig. 81, $P_0 = \frac{1}{4}R$; suppose P_f as found by experiment to be $\frac{5}{16}R$; what is the modulus of the combination? *Ans.* 80%.

In the following examples friction is neglected:

EXAMPLES.

1. In a system of six movable pulleys, of the kind described in Art. 80, what weight can be sustained by a power of 12 *lbs.*? *Ans.* 768 *lbs.*

2. In a combination of pulleys in two blocks, when there are six pulleys in each block, what weight can a power of 12 *lbs.* sustain in equilibrium? *Ans.* 144 *lbs.*

3. In a combination of separate movable pulleys, the resistance is 576 *lbs.*, and the power that keeps it in equilibrium is 9 *lbs.* How many pulleys in the combination? *Ans.* 6.

4. In a combination of pulleys in two blocks, with a single rope, the power is 62 *lbs.*, and the resistance 496 *lbs.* How many pulleys in each block? *Ans.* 4.

5. In a combination of two movable pulleys, the inclination of the ropes at each pulley is 120°. What is the power required to support a weight of 27 *lbs.* *Ans.* 9 *lbs.*

The Wheel and Axle.

82. The **wheel and axle** consists of a wheel, A, mounted on an axle, B. The power is applied at one extremity of a rope wrapped around the wheel, and the resistance at one extremity of a second rope, wrapped around the axle in a contrary direction. The whole is supported by pivots projecting from the ends of the axle. In deducing the conditions of equilibrium, we shall suppose the power and resistance to be at right angles to the axis, and that friction is neglected.

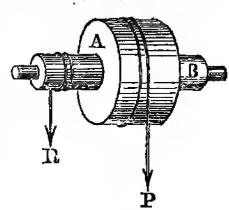

Fig. 82.

Denote the power by P, the resistance by R, the radius of the wheel by r, and the radius of the axle by r'. We shall have, in case of equilibrium (Art. 31),

$$Pr = Rr', \quad \text{or} \quad P : R :: r' : r.$$

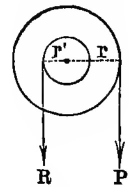

Fig. 83.

That is, *the power is to the resistance, as the radius of the axle, to the radius of the wheel.*

By suitably varying the dimensions of the wheel and axle, any amount of *mechanical advantage* may be obtained.

If we draw a line from the point of contact of the first rope with the wheel, to the point of contact of the second rope with the axle, the power and resistance being parallel, it will cut the axis of revolution at a point that divides the line through the points of contact into parts, inversely proportional to the power and resistance. This is the point of application of the resultant of these forces. The resultant is equal to the sum of the forces, and by the principal of moments, the pressure on each pivot may be computed. When the weight of the machine is taken into account, we regard it as a vertical force applied at the centre of gravity of the wheel and axle. The pressures on each pivot due to this weight may be computed separately, and the results combined with those already found.

The modulus is found as in the preceding article.

Combinations of Wheels and Axles.

83. If the rope of the first axle be passed around a second wheel, and the rope of the second axle around a third wheel, and so on, a combination will result, capable of affording great mechanical advantage. The figure represents a combination of two wheels and axles. To deduce the conditions

of equilibrium, denote the power by P, the resistance by R, the radius of the first wheel by r, that of the first axle by r', that of the second wheel by r'', and that of the second axle by r'''.

If we denote the tension of the connecting rope by t, this may be regarded as a power applied to the second wheel. From what was demonstrated for the wheel and axle, we shall have,

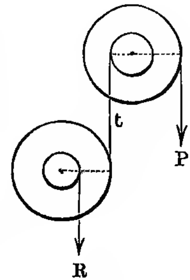

Fig. 84.

$$Pr = tr', \quad \text{and} \quad tr'' = Rr'''.$$

Multiplying these equations member by member, and reducing, we have,

$$Prr'' = Rr'r'''; \quad \text{or,} \quad P : R :: r'r''' : rr''.$$

In like manner, were there any number of wheels and axles in the combination, we might deduce the relation,

$$Prr''r^{\text{iv}} \ldots = Rr'r'''r^{\text{v}} \ldots;$$

$$\text{or,} \quad P : R :: r'r'''r^{\text{v}} \ldots\ldots : rr''r^{\text{iv}} \ldots\ldots \quad (84)$$

That is, *the power is to the resistance, as the continued product of the radii of the axles, to the continued product of the radii of the wheels.*

The principle just explained, is applicable to machinery in which motion is transmitted from wheel to wheel by bands, or belts. An endless band, called the driving belt, passes around one drum on the axle of the driving wheel, and around another on that of the driven wheel.

The Crank and Axle, or Windlass.—The Capstan.

84. The windlass consists of an **axle**, AB, and a **crank**, BCD. The power is applied to the **crank-handle**, DC, and the resistance to a rope wrapped around the axle. The distance, CB, from the handle to the axis, is the **crank-arm.**

The relation between the power and resistance is the same as in the wheel and axle, except that we substitute the crank-arm for the radius of the wheel.

Hence, *the power is to the resistance, as the radius of the axle, to the crank-arm.*

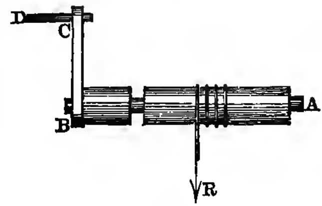

Fig. 85.

This machine is used in drawing water from wells, in raising ore from mines, and the like. It is also used in combination with other machines.

The **Capstan** differs in no material respect from the windlass except in having its axis vertical. The capstan consists of a vertical axle passing through guides, and having holes at its upper end for the insertion of levers. It is used on shipboard for raising anchors. The conditions of equilibrium are the same as in the windlass.

The Differential Windlass.

85. This differs from the common windlass in having its axle formed of two cylinders, *A* and *B*, of different diameters. A rope is attached to the larger cylinder, and wrapped several times around it, after which it passes round the movable pulley, *C*, and, returning, is wrapped in a contrary direction about the smaller cylinder, to which the second end of the rope is made fast. The power is applied at the crank-handle, *FE*, and the resistance to the hook of the movable pulley. When the crank is turned so as to wind the rope on the larger cylinder, it un-

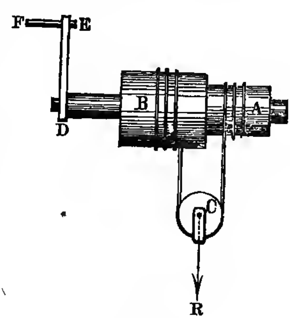

Fig. 86.

winds it from the smaller one, but in less degree, and the total effect is to raise the resistance, R.

The condition of equilibrium when friction is neglected may be found from the general theorem of work (Eq. 25). Denote the length of the crank-arm, DE, by c, the radius of the larger cylinder by r, and that of the smaller by r', and suppose P to act perpendicularly to DE. The path of the point of application of the power P during one turn of the windlass is $2\pi c$; the corresponding path of the resistance, R, is equal to one half the difference of the circumferences of the two cylinders or to $\pi(r - r')$; but these paths are to each other as the *elementary* paths of P and R, that is,

$$\frac{\delta p}{\delta r} = \frac{2\pi c}{\pi(r - r')} = \frac{2c}{r - r'}.$$

In this case equation (25) becomes $R\delta r - P\delta p = 0$; reducing and substituting as above, we have

$$\frac{R}{P} = \frac{2c}{r - r'}, \quad . \quad . \quad . \quad . \quad . \quad (85)$$

That is, *the power is to the resistance as the difference of the radii of the cylinders is to twice the length of the crank-arm.*

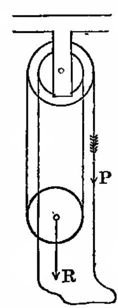

Fig. 87.

By increasing the crank-arm and diminishing the difference between the radii of the cylinders, any amount of mechanical advantage may be obtained; but the amount of rope required for a single turn is so great as to render the contrivance in the form described of little practical value. The principle of this windlass, finds an application in a machine known as WESTON'S pulley-block. In this combination, there are two pulleys nearly equal in size, and turning together as one in the upper block. An endless chain takes

the place of the rope, and is prevented from slipping by projecting pins. The power is applied to the portion of the chain that leaves the larger pulley, and the chain continues to run till the weight is raised.

To trace the course of the chain, let us commence at the point where it leaves the lower pulley: from this it ascends, passing around the larger pulley in the upper block; descending so as to leave a sufficient amount of slack, it again rises to the upper block, passes around the smaller pulley, and returns to the place of beginning.

To find the modulus of this machine apply a power P that is on the eve of overcoming the resistances: then divide the value of P as found from (85) by this result.

Wheel-work.

86. The principle employed in finding the relation between the power and resistance in a train of wheel-work is the same as that used in discussing the wheel and axle and its modifications.

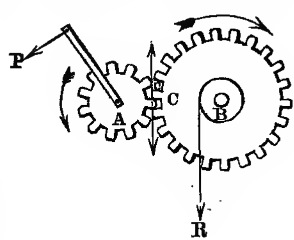

Fig. 88.

To illustrate, we take a case in which the power is applied to a crank-handle that is attached to the axis of a toothed wheel, A; the teeth of this wheel work into the spaces of the toothed wheel, B, and the resistance is attached to a rope wound round the arbor of the last wheel. In order that A may communicate motion to B, the number of teeth in their circumferences should be proportional to their radii, and the spaces between the teeth in one wheel should be wide enough to receive the teeth of the other, but not wide enough to allow much play. The teeth should always come in contact at the same distances from the centres

of the wheels, and those distances are taken as the radii of the wheels.

Denote the power by P, the resistance by R, the crank-arm by c, the radius of the wheel A by r, that of the wheel B by r', that of its arbor by r'', and suppose the power and resistance in equilibrium. The power tends to turn the wheels in the direction of the arrow-heads. This tendency is counteracted by the resistance which acts in a contrary direction. If we denote the pressure at C by R', we have, from what precedes,

$$Pc = R'r \quad \text{and} \quad R'r' = Rr'';$$

whence, by multiplication and reduction,

$$Pcr' = Rrr'', \quad \text{or,} \quad P : R :: rr'' : cr' \quad . \quad . \quad . \quad . \quad (86)$$

That is, *the power is to the resistance, as the continued product of the alternate arms of lever, beginning at the resistance, is to the continued product of the alternate arms of lever beginning at the power.*

When there are any number of wheels in the train between the power and resistance, we find a condition of equilibrium similar to that expressed by Equation 84.

Examples.

1. A power of 5 *lbs.*, acting at the circumference of a wheel whose radius is 5 feet, supports a resistance of 200 *lbs.*, applied at the circumference of the axle. What is the radius of the axle?

Ans. $1\frac{1}{2}$ inches.

2. The radius of the axle of a windlass is 3 inches, and the crank-arm 15 inches. What power must be applied to the crank-handle, to support a resistance of 180 *lbs.*, applied at the circumference of the axle?

Ans. 36 *lbs.*

3. A power, P, acts on a rope 2 inches in diameter, passing over a wheel whose radius is 3 feet, and supports a resistance of 320 *lbs.*, ap-

plied by a rope of the same diameter, passing over an axle whose radius is 4 inches. What is the value of P, the thickness of the rope being taken into account? *Ans.* $43\frac{9}{37}$ *lbs.*

The Screw.

87. The **screw** consists of a solid cylinder, enveloped by a spiral projection called **the thread.** The thread may be generated as follows: let an isosceles triangle be placed so that its base shall coincide with an element of the cylinder, and its plane pass through the axis. Let the triangle be revolved uniformly about the axis, and at the same time moved uniformly in the direction of the axis, at such a rate that it shall pass over a distance equal to the base of the triangle in one revolution. The solid generated by the triangle is the thread of the screw. The vertical distance traversed by the generatrix in one revolution is technically called the **distance between the threads.**

Fig. 89.

The screw just described has a **triangular thread.** Had we used a rectangle, instead of a triangle, and imposed the condition, that the motion in the direction of the axis during one revolution, should be twice its base, we should have had a screw with a **rectangular thread**, as in the figure.

The screw works into a piece called a **nut**, generated in a manner analogous to that just described, except that what is solid in the screw is hollow in the nut; it is, therefore, exactly adapted to receive the thread of the screw. Sometimes the screw is fast, and the nut turns on it; in this case, the nut has a motion of revolution, combined with a longitudinal motion. Sometimes the nut is fast, and the screw turns within it; in this case, the screw has a motion in the direction of its axis, in connection with a motion of rotation. The conditions of equilibrium are the same for each. In both cases, the power is applied at the extremity of a lever perpendicular to the axis of the screw.

We suppose the nut to be fixed and the screw to be movable. The mechanical advantage may then be found by the general theorem of work, friction being neglected.

Let the length of the lever be denoted by l, and the distance between the threads by d; the path of the power P in one revolution will be $2\pi l$ and the corresponding path of the resistance R will be d: but these paths are to each other as the elementary paths of P and R, that is,

$$\frac{\delta p}{\delta r} = \frac{2\pi l}{d}.$$

Hence, as in Art. 85, we have

$$\frac{R}{P} = \frac{2\pi l}{d} \quad . \quad . \quad . \quad . \quad . \quad (87)$$

That is, *the power is to the resistance, as the distance between the threads, is to the circumference described by the point of application of the power.*

By diminishing the distance between the threads, any amount of mechanical advantage may be obtained; too great a diminution will weaken the threads which will then be liable to be *stripped* from the cylinder. This objection may be overcome, in part, by using the differential screw.

The Differential Screw.

88. The **differential screw** consists of an ordinary screw, into the end of which works a smaller screw, having its axis coincident with the first. The distance between the threads of the second screw is less than that between those of the first, and this difference may be made as small as desirable. The second screw is so arranged that it admits of longitudinal motion, but not of rotation.

When the lever is turned once around the outer screw moves downward a distance d, equal to the distance between its threads; at the same time the inner screw is drawn up a distance equal to, d', the distance between *its* threads. Hence,

the aggregate movement *downward* is equal to the difference of the distances between the threads of the two screws : this is the path of the resistance R. The path of the power is the same as before ; hence, as in the last article, we have

$$\frac{R}{P} = \frac{2\pi l}{d-d'} \quad . \quad . \quad . \quad . \quad . \quad (88)$$

Hence, *the power is to the resistance, as the difference of the distances between the threads of the two screws, is to the circumference described by the point of application of the power.*

The modulus in this and in the preceding case might be found in the same manner as in the case of the inclined plane. In either case it will be very small on account of friction, which is very great. It is better in both cases to find the modulus by experiment. The value of P, as found from (87) or (88), is divided by that value of P which is found just capable of overcoming all the resistances. The latter value may be determined by the aid of a dynamometer.

The Endless Screw.

89. The **endless screw** is a screw secured by shoulders, so that it cannot be moved longitudinally, and working into a toothed wheel. The distance between the teeth is nearly the same as the distance between the threads of the screw. When the screw is turned, it imparts a rotary motion to the wheel, which may be utilized by any mechanical device. The conditions of equilibrium are the same as for the screw, the resistance in this case being offered by the wheel, in the direction of its circumference.

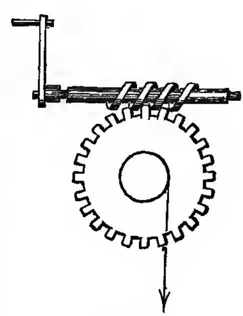

Fig. 90.

Examples.

1. What must be the distance between the threads of a screw, that a power of 28 *lbs.*, acting at the extremity of a lever 25 inches long, may sustain a weight of 10,000 *lbs.*? *Ans.* .4396 inches.

2. The distance between the threads of a screw is $\frac{1}{8}$ of an inch. What resistance can be supported by a power of 60 *lbs.*, acting at the extremity of a lever 15 inches long? *Ans.* 16,964 *lbs.*

3. The distance from the axis of the trunnions of a gun weighing 2,016 *lbs.* to the elevating screw is 3 feet, and the distance of the centre of gravity of the gun from the same axis is 4 inches. If the distance between the threads of the screw be $\frac{2}{8}$ of an inch, and the length of the lever 5 inches, what power must be applied to sustain the gun in a horizontal position? *Ans.* 4.754 *lbs.*

The Wedge.

90. The **wedge** is a combination of two inclined planes. It is bounded by a rectangle, BD, called the **back**; two rectangles, AF and DF, called **faces**; and two isosceles triangles, called **ends**. The line, EF, in which the faces meet, is the **edge**.

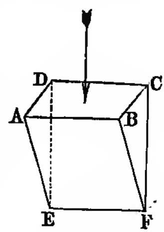

Fig. 91.

The power is applied at the back, to which it should be normal, and the resistance is applied to the faces, and normal to them. One half the resistance is applied to one face, and the other half to the other face. Let ABC be a section of a wedge by a plane at right angles to the edge. Denote the power by P, the resistance opposed to each face by $\frac{1}{2}R$, and the angle BAC by 2ϕ. Produce the directions of the resistances till they intersect in O. This point will be on the line of the direction of the power. Because the three forces P, $\frac{1}{2}R$, and $\frac{1}{2}R$ are in equilibrium, we have,

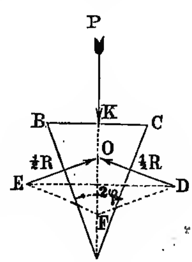

Fig. 92.

$$P : \tfrac{1}{2}R :: \sin EOD : \sin POD.$$

But, DO and EO are perpendicular to AC and AB; hence,

$$\sin EOD = \sin 2\phi = 2 \sin\phi \cos\phi.$$

In like manner, PO and DO are perpendicular to KC and AC; hence,

$$\sin POD = \sin ACK = \cos\phi.$$

Substituting, and reducing, we have,

$$P : \tfrac{1}{2}R :: 2 \sin\phi : 1,$$

or, $$P : R :: KC : AC \quad . \quad . \quad . \quad . \quad . \quad (89)$$

That is, *the power is to the resistance, as half the breadth of the back, is to the length of the face.*

The mechanical advantage of the wedge may be increased by diminishing the breadth of the back, or, in other words, by making the edge sharper. The principle of the wedge finds an application in cutting instruments. By diminishing the thickness of the back, the instrument is weakened; hence the necessity of forming cutting instruments of hard and tenacious material.

V.—KINETICS.

1° Rectilinear Motion.

Motion.

91. A material point is **in motion** when it continually changes its position with respect to objects that we regard as *fixed.* If the path of a moving point is a straight line, the motion is *rectilinear;* if it is a curved line, the motion is *curvilinear.* When the motion is curvilinear, we may regard the path as made up of infinitely short straight lines; that is, we may consider it as a polygon, whose sides are infinitely small. If any side of this polygon be prolonged in the direction of the motion, it will be a tangent to the curve. Hence, we say, that *a point always moves in the direction of a tangent to its path.*

Fundamental Formulas.

92. We have seen (Eq. 4), that in *uniform* motion the velocity is equal to the space described in any time divided by the time: but any kind of motion may be regarded as uniform for the infinitesimal time dt; hence, if we denote the space described in that time by ds, we shall have, for any kind of motion,

$$v = \frac{ds}{dt} \quad . \quad . \quad . \quad . \quad . \quad (90)$$

The **acceleration** due to a force is the rate at which it can *impart,* or **generate** velocity. If we denote the acceleration due to a force by ϕ and the velocity it can generate in the time dt by dv, we have, from the definition,

$$\phi = \frac{dv}{dt} \quad . \quad . \quad . \quad . \quad . \quad (91)$$

Differentiating (90) and substituting in (91), we have

$$\phi = \frac{d^2s}{dt^2} \quad . \quad . \quad . \quad . \quad . \quad (92)$$

The measure of an incessant force F (Art. 11), is the rate at which it can generate momentum; hence, if we denote the mass on which the force acts by m we shall have

$$F = m\phi = m\frac{d^2s}{dt^2} \quad . \quad . \quad . \quad . \quad . \quad (93)$$

It is evident that F acts in the direction of the motion. It is called the **moving force.** When a body moves upon any plane curve the motion may be regarded as taking place in the direction of two rectangular axes. If we denote the effective components of the moving force in the direction of these axes, by X and Y, the spaces described being denoted by x and y, we shall have, from the second law of Newton (Art. 12),

$$X = m\frac{d^2x}{dt^2}, \qquad Y = m\frac{d^2y}{dt^2} \quad . \quad . \quad . \quad . \quad . \quad (94)$$

Uniformly Varied Motion.

93. In uniformly varied motion the velocity increases (or diminishes) *uniformly*, that is, the velocity generated in the time dt is always the same: hence, the *acceleration is constant.*

Denoting the constant acceleration by f, and supposing the mass acted upon to be the unit of mass, we have, from Equation (93),

$$\frac{d^2s}{dt^2} = f \quad . \quad . \quad . \quad . \quad . \quad (94)$$

Multiplying by dt, and integrating, we have,

$$\frac{ds}{dt} = ft + C \quad . \quad . \quad . \quad . \quad . \quad (95)$$

Multiplying again by dt, and integrating, we have,

$$s = \tfrac{1}{2}ft^2 + Ct + C' \quad . \quad . \quad . \quad . \quad . \quad (96)$$

In accordance with the principle of Integral Calculus, that the arbitrary constant added to an integral is a quantity of the same kind as the integral, C, in (95), must be a *velocity*, and C', in (96), must be a *space*. Denoting these by v' and s' respectively, equations (95) and (96) may be written,

$$v = v' + ft \,; \quad . \quad . \quad . \quad . \quad . \quad (97)$$

$$s = s' + v't + \tfrac{1}{2}ft^2 \quad . \quad . \quad . \quad . \quad . \quad (98)$$

If we make $t = 0$, we have, $v = v'$ and $s = s'$, that is, v' and s' are the velocity and the space at the beginning of the time t; the former is called the **initial velocity**, and the latter the **initial space.**

Equations (97) and (98) are the fundamental equations of uniformly varied motion.

If we suppose the body to move from *rest* at the beginning of the time t; both v' and s' are 0, and equations (97) and (98) become

$$v = ft \,; \quad . \quad . \quad . \quad . \quad . \quad (99)$$

$$s = \tfrac{1}{2}ft^2 \quad . \quad . \quad . \quad . \quad . \quad (100)$$

From the first of these we see that *the velocity varies as the time;* and from the second we see that *the space varies as the square of the time.*

If, in (100), we make $t = 1$, we have,

$$s = \tfrac{1}{2}f, \quad \text{or} \quad f = 2s \quad . \quad . \quad . \quad . \quad . \quad (101)$$

That is, *when a body moves from rest, under the action*

of a constant force, the acceleration is equal to twice the space passed over in the first second of time.

If we suppose f to be *positive*, the motion is **uniformly accelerated**; if we suppose f to be *negative*, the motion is **uniformly retarded.** In the latter case equations (97) and (98) become

$$v = v' - ft\,; \quad . \; . \; . \; . \; . \quad (102)$$

$$s = s' + v't - \tfrac{1}{2}ft^2 \quad . \; . \; . \; . \; . \quad (103)$$

Application to Falling Bodies.

94. The **force of gravity** is the force exerted by the earth upon all bodies exterior to it, tending to draw them toward it. It is found by observation that this force is sensibly *directed toward the centre of the earth*, and that its intensity *varies inversely as the square of the distance from the centre.*

Because the centre of the earth is so distant from the surface, the variation in intensity for small elevations above the surface is inappreciable. Hence, we may regard the force of gravity at any place on, or near, the earth's surface as constant. The force of gravity acts equally on all the particles of a body, and were there no resistance offered, it would impart the same velocity, in the same time, to any two bodies whatever. The atmosphere, however, offers a resistance, which tends to retard the motion of bodies falling through it; and of two bodies of equal mass, it retards that one most, which presents the greatest surface to the direction of the motion. In discussing the laws of falling bodies, it will be found convenient to regard them as being in vacuo, and in this case the equations of the preceding article are immediately applicable. The effects of atmospheric resistance may be taken into account, as corrections, or in certain cases the

motions may be made so slow that their effects may be neglected.

If we denote the acceleration due to gravity by g, and the space fallen through by h, both being supposed position downward; and, furthermore, if we suppose the body to start from rest at the beginning of the time t, we have, from (99) and (100),

$$v = gt; \quad . \quad . \quad . \quad . \quad . \quad (104)$$

$$h = \tfrac{1}{2}gt^2 \quad . \quad . \quad . \quad . \quad . \quad (105)$$

That is, *the velocities are proportional to the times, and the spaces to the squares of the times.*

The value of g in the latitude of New York is not far from $32\tfrac{1}{6}$ *ft.*, or to the nearest *tenth of a foot* it may be called 32.2 *ft.*, which is sufficiently accurate for ordinary computations.

From equation (105), we have,

$$t = \sqrt{\frac{2h}{g}} \quad . \quad . \quad . \quad . \quad . \quad (106)$$

That is, *the number of seconds required for a body to fall through any height is equal to the square root of the quotient obtained by dividing twice the height in feet by the acceleration due to gravity.*

Substituting this value of t in (104) and reducing, we have,

$$v = \sqrt{2gh} \quad . \quad . \quad . \quad . \quad . \quad (107)$$

from which, we have,

$$h = \frac{v^2}{2g} \quad . \quad . \quad . \quad . \quad . \quad (108)$$

In equations (107) and (108), v is called **the velocity due to the height** h, and h is called **the height due to the velocity** v.

If the body be projected downward with a velocity v', the circumstances of motion will be made known by the equations,

$$\left.\begin{array}{l} v = v' + gt, \\ h = v't + \frac{1}{2}gt^2. \end{array}\right\} \quad . \; . \; . \; . \; . \quad (109)$$

In these equations, the **origin of spaces** is taken at the point from which the body is projected downward.

Motion of Bodies Projected Vertically Upward.

95. Suppose a body projected vertically upward from the origin of spaces, with a velocity v', and afterward acted on by the force of gravity. In this case, the force of gravity acts to retard the motion. Making, in (97) and (98), $s' = 0$, $f = -g$, and $s = h$, they become,

$$v = v' - gt \quad . \; . \; . \; . \; . \quad (110)$$

$$h = v't - \tfrac{1}{2}gt^2 \quad . \; . \; . \; . \quad (111)$$

In these equations h is positive upward, and negative downward.

From equation (110), we see that the velocity diminishes as the time increases. The velocity is 0, when,

$$v' - gt = 0, \text{ or when } t = \frac{v'}{g}. \quad . \; . \; . \; . \; . \; . \quad (112)$$

When t is greater than $\frac{v'}{g}$, v is negative, and the body retraces its path: hence, *the time required for the body to reach its highest elevation, is equal to the initial velocity, divided by the force of gravity.*

Eliminating t, from (110) and (111), we have,

$$h = \frac{v'^2 - v^2}{2g} \quad . \; . \; . \; . \; . \quad (113)$$

Making $v = 0$, in the last equation, we have,

$$h = \frac{v'^2}{2g} \quad . \; . \; . \; . \; . \quad (114)$$

Hence, *the greatest height to which the body will*

ascend, is equal to the square of the initial velocity, divided by twice the force of gravity.

This height is that due to the initial velocity (Eq. 108).

If, in (110), we make $t = \frac{v'}{g} - t'$, we find,

$$v = gt' \quad . \quad . \quad . \quad . \quad . \quad (115)$$

If, in the same equation, we make $t = \frac{v'}{g} + t'$, we find,

$$v = -\, gt' \quad . \quad . \quad . \quad . \quad . \quad (116)$$

Hence, *the velocities at equal times before and after reaching the highest points are equal.*

The difference of signs shows that the body is moving in opposite directions at the times considered.

If we substitute these values of v successively, in (113), we find in both cases,

$$h = \frac{v'^2 - g^2t'^2}{2g}; \quad . \quad . \quad . \quad . \quad . \quad (117)$$

hence, the points at which the velocities are equal, in ascending and descending, are equally distant from the highest point; that is, they are coincident. Hence, *if a body be projected vertically upward, it will ascend to a certain point, and then return upon its path, in such manner, that the velocities in ascending and descending are equal at the same points.*

EXAMPLES.

1. Through what distance will a body fall from rest in a vacuum, in 10 seconds, and through what space will it fall during the last second?

Ans. $1608\frac{1}{3}$ *ft.*, and $305\frac{1}{2}$ *ft.*

2. In what time will a body fall from rest through 1200 feet?

Ans. 8.63 *sec.*

3. A body was observed to fall through a height of 100 feet in the last second. How long was the body falling, and through what distance did it descend?

Ans. $t = 3.6$; $s = 208.4$ *ft.*

4. A body falls through 300 feet. Through what distance does it fall in the last two seconds ?

Ans. The entire time occupied is 4.32 seconds. The distance fallen through in 2.32 *sec.*, is 86.57 *ft.* Hence, the distance required is 300 *ft.* — 86.57 *ft.* = 213.43 *ft.*

5. A body is projected upward, with a velocity of 60 feet. To what height will it rise ? *Ans.* 55.9 *ft.*

6. A body is projected upward, with a velocity of 483 *ft.* In what time will it rise 1610 feet ?

We have, from equation (111),

$$1610 = 483t - 16\tfrac{1}{12}t^2 \,; \qquad \therefore \quad t = \tfrac{2898}{193} \pm \tfrac{2161}{193} \,;$$

$$\text{or,} \quad t = 26.2 \; sec., \quad \text{and} \quad t = 3.82 \; sec.$$

The smaller value of t gives the time required ; the larger value gives the time occupied in rising to its greatest height, and returning to the point 1610 feet from the starting-point.

7. A body is projected upward, with a velocity of 161 feet, from a point $214\frac{2}{3}$ feet above the earth. In what time will it reach the earth, and with what velocity will it strike ? *Ans.* 11.2 *sec.*; 199 *ft.*

8. Suppose a body to have fallen through 50 feet, when a second begins to fall just 100 feet below it. How far will the latter body fall before it is overtaken by the former ? *Ans.* 50 feet.

9. Suppose a body to descend from rest for $3\frac{1}{2}$ seconds, and then to move uniformly for $2\frac{1}{2}$ seconds with the acquired velocity ; what is the entire distance passed over ? *Ans.* 478 *ft. nearly.*

10. A body falls from a height of 400 *ft.* at the same instant that a body is projected upward from the earth with a velocity of 100 *ft.* ; at what height above the earth will they pass each other ? *Ans.* $142\frac{2}{3}$ *ft.*

11. With what velocity must a body be projected vertically upward, that it may rise to a height of 210 *ft.* in 3 *seconds ?* *Ans.* $118\frac{1}{4}$ *ft.*

Restrained Vertical Motion.

96. We have seen (Eq. 93) that the measure of a moving force is equal to the acceleration due to the force *multiplied* by the mass moved ; hence, conversely, the acceleration due to a moving force is equal to the force *divided* by the mass moved. Consequently, in case of a body falling freely, the

moving force varies directly as the mass moved, the acceleration g being constant. If, however, we increase the mass moved, without changing the moving force, we shall correspondingly diminish the acceleration.

To show how this may be done, let A be a fixed pulley, mounted on a horizontal axis, and W and W', unequal weights attached to the extremities of a flexible cord passing over the pulley. If the weight, W, be greater than W', the former will descend, and draw the latter up.

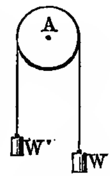

Fig. 93.

In this case, the moving force is the difference of the weights, W and W'; the mass moved is the sum of the masses of W and W', together with that of the pulley and connecting cord. The different parts of the pulley move with different velocities, but the effect of its mass may be replaced by that of some other mass at the circumference of the pulley. Denoting this mass, together with the mass of the cord, by m'', and the masses of W and W' by m and m', we have the entire mass moved equal to $m + m' + m''$, and for the moving force we have $(m - m')g$; hence, the acceleration, denoted by g', is given by the equation,

$$g' = \frac{m - m'}{m + m' + m''} g. \quad . \quad . \quad . \quad . \quad . \quad (118)$$

This force being constant, the motion produced by it is *uniformly varied*, and the circumstances of that motion will be made known by substituting the above expression for f, in equations (97) and (98).

Examples.

1. Two weights of 5 *lbs.* and 4 *lbs.* are suspended from the extremities of a cord passing over a fixed pulley, the weight of the pulley and cord being neglected. What is the acceleration, what distance will each weight describe in the first second, and what is the tension of the cord?

Ans. $g' = 3.574\,ft.$; $s = 1.787\,ft.$ To find the tension of the cord denoted by t: we have the moving force acting on the heavier body equal to $(5 - t)\,g$, and the acceleration due to this force is $\left(\frac{5-t}{5}\right)g$; the moving force acting on the lighter body is $(t - 4)\,g$, and the corresponding acceleration is $\left(\frac{t-4}{4}\right)g$; equating these accelerations, and solving, we have, $t = 4\frac{4}{9}\,lbs.$

2. A weight of 1 *lb.*, hanging over a pulley, descends and drags a second weight of 5 *lbs.* along a horizontal plane. Neglecting the mass of the pulley and all hurtful resistances, to what will the acceleration be equal, and through what space will the descending body move in the first second? *Ans.* $g' = 5.3622\,ft.$; $s = 2.6811\,ft.$

3. Two bodies, each weighing 5 *lbs.*, are attached to a string passing over a fixed pulley. What distance will each body move in 10 seconds, when a pound weight is added to one of them, and what velocity will have been generated at the end of that time, all hurtful resistances being neglected? *Ans.* $s = 146.2\,ft.$; $v = 29.24\,ft.$

4. Two weights, of 16 *oz.* each, are attached to the ends of a string passing over a fixed pulley. What weight must be added to one of them, that it may descend through a foot in two seconds, hurtful resistances being neglected?

Ans. The required weight being w, we have, $g' = \frac{w}{32 + w}g$; but $s = \frac{1}{2}g't^2$, which for $s = 1$, and $t = 2$, gives, $w = 0.505\,oz.$

Atwood's Machine.

97. Atwood's machine is a contrivance to illustrate the laws of falling bodies. It consists of a vertical post, *AB*, about 10 feet in height, supporting, at its upper extremity, a fixed pulley, *A*. To obviate, as far as possible, the resistance of friction, the axle is made to turn on friction rollers. A silk string passes over the pulley, and at its extremities are fastened two equal weights, *C* and *D*. To impart motion to the weights, a small weight, *G*, in the form of a bar, is laid on *C*, and by diminishing its mass, the acceleration may be rendered as small as desirable.

The rod, *AB*, graduated to feet and decimals, is provided

with sliding stages, E and F; the upper one is in the form of a ring, which will permit C to pass, but not G; the lower one is in the form of a plate, which is intended to intercept the weight C. Connected with the instrument is a seconds pendulum for measuring time.

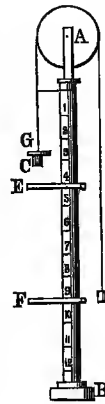

Fig. 94.

Suppose the weights, C and D, each equal to 150 grains, the weight of the bar 24 grains, and let a weight of 62 grains at the circumference of the pulley, produce the same resistance by its inertia as that actually produced by the pulley and cord. Then will the fraction $\frac{m - m'}{m + m' + m''}$, become equal to $\frac{24}{386}$; and this, multiplied by $32\frac{1}{6}$, gives $g' = 2$. This value, substituted for g, in (104) and (105), gives,

$$v = 2t, \quad \text{and} \quad h = t^2.$$

If, in these equations, we make $t = 1$ *sec.*, we have $h = 1$, and $v = 2$. If we make $t = 2$ *sec.*, we, in like manner, have $h = 4$, and $v = 4$. If we make $t = 3$ *sec.*, we have $h = 9$, and $v = 6$, and so on. To verify these results experimentally, commencing with the first :—The weight, C, is drawn up till it comes opposite the 0 of the graduated scale, and the bar G is placed on it. The weight thus set is held in its place by a spring. The ring, E, is set at 1 foot, and the stage, F, at 3 feet from the 0. When the pendulum reaches one of its extreme limits, the spring is pressed back, the weight, CG, descends, and as the pendulum completes its vibration, the bar G strikes the ring, and is retained. The acceleration then becomes 0, and C moves on uniformly, with the velocity acquired, during the first second; and it will be observed that C strikes the second stage just as the pendulum completes its second vibration.

Had *F* been set at 5 feet from the 0, *C* would have reached it at the end of the third vibration of the pendulum. Had it been 7 feet from the 0, it would have reached it at the end of the fourth vibration, and so on.

To verify the next result, we set the ring, *E*, at 4 feet from the 0, and the stage, *F*, at 8 feet from the 0, and proceed as before. The ring will intercept the bar at the end of the second vibration, and the weight will strike the stage at the end of the third vibration, and so on.

By making the weight of the bar less than 24 grains, the acceleration is diminished, and, consequently, the spaces and velocities, correspondingly diminished. The results may be verified as before.

Motion of Bodies on Inclined Planes.

98. If a body is placed on an inclined plane, and abandoned to the action of its own weight, it will either slide or roll down the plane, provided there be no friction between it and the plane. If the body is spherical, it will roll, and in this case friction may be disregarded. Let the weight of the body be resolved into two components, one perpendicular to the plane, and the other parallel to it: the plane of these components will be vertical, and also perpendicular to the given plane. The effect of the first component will be counteracted by the resistance of the plane, whilst the second will act as a constant force, urging the body down the plane. The force being constant, the body will have a uniformly varied motion, and equations (97) and (98) will be applicable. The acceleration may be found by projecting the acceleration due to gravity on the inclined plane.

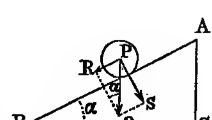

Fig. 95.

Let *AB* represent the inclined plane, and *P* the centre of gravity of a body resting on it. Let *PQ* represent the

force of gravity, denoted by g, and PR its component, parallel to AB, PS being the normal component.

Denote PR by g', and the angle ABC by α. Then, since PQ is perpendicular to BC, and QR to AB, the angle, RQP, is equal to ABC, or to α. From the right-angled triangle, PQR, we have,

$$g' = g \sin \alpha.$$

But the triangle, ABC, is right-angled, and, if we denote its height, AC, by h, and its length, AB, by l, we shall have $\sin \alpha = \frac{h}{l}$, which, being substituted above, gives,

$$g' = \frac{gh}{l} \quad . \quad . \quad . \quad . \quad . \quad (119)$$

This value of g' is the acceleration due to the moving force. Substituting it for f, in equations (97) and (98), we have,

$$v = v' + \frac{gh}{l}t, \quad . \quad . \quad . \quad . \quad . \quad (119)$$

$$s = s' + v't + \frac{gh}{2l}t^2 \quad . \quad . \quad . \quad . \quad . \quad (120)$$

If the body start from rest at A, taken as the origin of spaces, then will $v' = 0$, and $s' = 0$, giving,

$$v = \frac{gh}{l}t \quad . \quad . \quad . \quad . \quad . \quad (121)$$

$$s = \frac{gh}{2l}t^2 \quad . \quad . \quad . \quad . \quad . \quad (122)$$

To find the time required for a body to move from the top to the bottom of the plane, make $s = l$, in (122); there will result,

$$l = \frac{gh}{2l}t^2; \ \therefore \ t = l\sqrt{\frac{2}{gh}} \quad . \quad . \quad . \quad . \quad . \quad (123)$$

Hence, *the time varies directly as the length, and inversely as the square root of the height.*

For planes having the same height, but different lengths, the radical factor of the value of t remains constant. Hence, *the times required for a body to move down planes having the same height, are to each other as their lengths.*

To determine the velocity with which a body reaches the bottom of the plane, substitute for t, in equation (121), *its value taken from equation* (123). We have, after reduction,

$$v = \sqrt{2gh}.$$

But this is the velocity due to the height h, Eq. 107. Hence, *the velocity generated in a body whilst moving down an inclined plane, is equal to that generated in falling through the height of the plane.*

Let O be the centre of a circle in a vertical plane, and let AB be its vertical diameter. Through A draw any chord AC and regard it as an inclined plane; also draw CD perpendicular to AB. Denote AC by l, AD by h, and AB by $2r$. The time required for a body to roll down AC is, from Eq. (123).

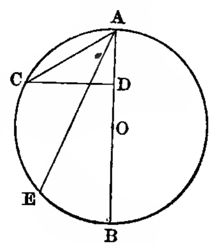

Fig. 96.

$$t = l\sqrt{\frac{2}{gh}}.$$

But from a property of the circle, we have

$$l = \sqrt{h \times 2r}.$$

Substituting this in the preceding equation, we have after reduction,

$$t = \sqrt{\frac{2\,(2r)}{g}} = \sqrt{\frac{2\,(AB)}{g}} \quad . \quad . \quad . \quad . \quad . \quad (124)$$

From (106) we see that this is the time required for a body to fall through the height AB. Hence, the time required for

a body to roll down any chord through A is equal to the time required for the body to fall through the vertical diameter; in other words, the time required for a body to roll down any chord passing through the point A is *constant.*

EXAMPLES.

1. An inclined plane is 10 feet long and 1 foot high. How long will it take a body to roll from top to bottom, and what velocity will it acquire. *Ans.* $t = 2\frac{1}{2}$ *sec.* nearly; $v = 8.02$ *ft.*

2. How far will a body descend from rest in 4 seconds, on an inclined plane whose length is 400 feet, and whose height is 300 feet? *Ans.* 193 *ft.*

3. How long will it take a body to descend 100 feet on a plane whose length is 150 feet, and whose height is 60? *Ans.* 3.9 *sec.*

4. There is a track, $2\frac{1}{2}$ miles in length, whose inclination is 1 in 35. What velocity will a car attain, in running the length of the road, by its own weight, hurtful resistances being neglected? *Ans.* 155.75 *ft.*, or, 106.2 *m.* per hour.

5. A railway train, having a velocity of 45 miles per hour, is detached from the locomotive on an ascending grade of 1 in 200. How far, and for what time, will the train continue to ascend the inclined plane? *Ans.* Distance = 13,540 *ft.*; $t = 6m.$ 50.3 *sec.*

6. A body weighing 5 *lbs.* descends vertically, and draws a weight of 6 *lbs.* up an inclined plane of 45°. How far will the first body descend in 10 seconds? *Ans.* 3.44 *ft.*

Body Falling in a Resisting Medium.

99. In the cases hitherto considered, we have supposed the motion to take place in vacuo: if a body falls through a resisting medium, like the atmosphere, it will experience a resistance which is generally assumed to vary as the square of the velocity. This resistance acts to diminish the acceleration due to gravity; hence, the acceleration thus diminished is expressed by the equation,

$$\phi = g - nv^2,$$

in which n is a constant whose value is to be determined by

experiment. Replacing ϕ by its value given in (91), and putting n, for convenience, equal to $\frac{g}{k^2}$, we have, after reduction,

$$\frac{dv}{dt} = g\left(\frac{k^2 - v^2}{k^2}\right), \quad . \quad . \quad . \quad . \quad . \quad (125)$$

from which, we find,

$$\frac{g}{k^2}\,dt = \frac{dv}{k^2 - v^2}.$$

Integrating (Calc. p. 119), and supposing $v = 0$ when $t = 0$, we have, after reduction,

$$\frac{2gt}{k} = l\left(\frac{k + v}{k - v}\right) \quad . \quad . \quad . \quad . \quad . \quad (126)$$

Passing to numbers, equation (126) becomes,

$$e^{\frac{2gt}{k}} = \frac{k + v}{k - v}, \quad . \quad . \quad . \quad . \quad . \quad (127)$$

which gives the relation between the time and the velocity.

Solving (127), with respect to v, we have,

$$v = k\left(\frac{e^{\frac{2gt}{k}} - 1}{e^{\frac{2gt}{k}} + 1}\right) \quad . \quad . \quad . \quad . \quad . \quad (128)$$

Dividing both terms of the second member by $e^{\frac{gt}{k}}$, replacing the first member by its value $\frac{ds}{dt}$, and multiplying through by dt, we have,

$$ds = \frac{k\left(e^{\frac{gt}{k}} - e^{-\frac{gt}{k}}\right)dt}{e^{\frac{gt}{k}} + e^{-\frac{gt}{k}}} \quad . \quad . \quad . \quad . \quad . \quad (129)$$

In the second member of (129), the numerator is equal to

the differential of the denominator multiplied by $\frac{k^2}{g}$; integrating, we have,

$$s = \frac{k^2}{g}\left[l\left(e^{\frac{gt}{k}} + e^{-\frac{gt}{k}}\right) + C\right] \quad . \quad . \quad . \quad . \quad . \quad (130)$$

To find C, suppose $s = 0$ when $t = 0$; we then have, $C = -l2$. Substituting in (130), we have, finally,

$$s = \frac{k^2}{g}l\left[\tfrac{1}{2}\left(e^{\frac{gt}{k}} + e^{-\frac{gt}{k}}\right)\right] \quad . \quad . \quad . \quad . \quad . \quad (131)$$

which expresses the relation between s and t when the body falls from rest at the beginning of the time t.

Body Projected into a Resisting Medium.

100. Let a body be projected into a resisting medium and suppose that it is not afterward acted upon by any other force than the resistance of the medium. Denote the velocity at the distance s from the point of projection by v and the resistance when the velocity is 1 by k. The acceleration will be negative and from what precedes, we have,

$$\frac{dv}{dt} = -kv^2, \quad \text{or,} \quad dt = -\frac{1}{k}\frac{dv}{v^2} \quad . \quad . \quad . \quad . \quad . \quad (132)$$

Integrating (132), we have,

$$t = \frac{1}{k} \cdot \frac{1}{v} + C \quad . \quad . \quad . \quad . \quad . \quad (133)$$

To find C, let $v = v_0$ when $t = 0$; this gives $C = -\frac{1}{k} \cdot \frac{1}{v_0}$, which in (133), gives

$$t = \frac{1}{k}\left(\frac{1}{v} - \frac{1}{v_0}\right) \quad . \quad . \quad . \quad . \quad . \quad (134)$$

which shows the relation between t and v.

From (134), we have

$$\frac{1}{v} = kt + \frac{1}{v_0}.$$

Substituting for v its value $\frac{ds}{dt}$, and solving, we have

$$ds = \frac{dt}{kt + \frac{1}{v_0}}, \therefore\ s = \frac{1}{k}l\left(kt + \frac{1}{v_0}\right) + C \quad . \quad . \quad . \quad . \quad . \quad (135)$$

Making $s = 0$ when $t = 0$, we find $C = -\frac{1}{k}l\left(\frac{1}{v_0}\right)$, hence (135), becomes,

$$s = \frac{1}{k}l\,(v_0kt + 1), \quad . \quad . \quad . \quad . \quad . \quad (136)$$

Which shows the relation between the space s and the time t.

Falling Bodies when Gravity is Variable.

101. In accordance with the Newtonian law, the attraction exerted by the earth on a body at different distances varies inversely as the squares of those distances. Denoting the radius of the earth, supposed a sphere, by r, the force of gravity at the surface by g, any distance from the centre greater than r, by s, and the force of gravity at that distance by ϕ, we have

$$s^2 : r^2 :: g : \phi; \quad \therefore \phi = \frac{gr^2}{s^2}.$$

Substituting this value of ϕ in (92), and at the same time making it negative, because it acts in the direction of s negative, we have

$$\frac{d^2s}{dt^2} = -\frac{gr^2}{s^2} \quad . \quad . \quad . \quad . \quad . \quad (137)$$

Multiplying by $2ds$, and integrating both members, we have,

$$\frac{ds^2}{dt^2} = \frac{2gr^2}{s} + C; \quad \text{or,} \quad v^2 = \frac{2gr^2}{s} + C \quad . \quad . \quad . \quad . \quad . \quad (138)$$

If we make $v = 0$, when $s = h$, we have

$$0 = \frac{2gr^2}{h} + C; \quad \therefore C = -\frac{2gr^2}{h},$$

$$\text{and, } v^2 = 2gr^2\left(\frac{1}{s} - \frac{1}{h}\right) \quad . \quad . \quad . \quad . \quad . \quad (139)$$

Equation (139) gives the velocity generated whilst the body is falling from the height h to the height s. If we make $h = \infty$, and $s = r$, (139) becomes,

$$v^2 = 2gr; \quad \therefore v = \sqrt{2gr} \quad . \quad . \quad . \quad . \quad . \quad (140)$$

In this equation the resistance of the air is not considered. If we make $g = 32.088$ *ft.*, and $r = 20{,}923{,}596$ *ft.*, their equatorial values, we find,

$$v = 36{,}664 \text{ ft., or nearly 7 miles.}$$

That is, *if a body were to fall from an infinite distance to the surface of the earth, its terminal velocity would be nearly 7 miles per second.*

Equation (140) enables us to compute the terminal velocity of a body falling from an infinite distance to the sun. In this case $g = 890.16$ *ft.*, and $r = 430{,}854.5$ miles, and the corresponding terminal velocity is,

$$v = 381 \text{ miles per second.}$$

To find the time required for a body to fall through any space, substitute $\frac{ds}{dt}$ for v in (139), and solve the result with respect to dt; this gives,

$$dt = -\sqrt{\frac{h}{2gr^2}} \cdot \frac{ds\sqrt{s}}{\sqrt{h-s}} = -\sqrt{\frac{h}{2gr^2}} \cdot \frac{sds}{\sqrt{hs-s^2}} \quad . \; . \; . \quad (141)$$

The negative sign is taken because s decreases as t increases. Reducing (141) by Formula E, and integrating by Formula [26], Calculus, we find,

$$t = \sqrt{\frac{h}{2gr^2}}\left[(hs-s^2)^{\frac{1}{2}} - \frac{h}{2}\text{versin}^{-1}\frac{2s}{h}\right] + C \quad . \; . \; . \; . \quad (142)$$

If $t = 0$ when $s = h$, we have,

$$C = \sqrt{\frac{h}{2gr^2}} \times \frac{\pi h}{2}.$$

This, in (142), gives,

$$t = \sqrt{\frac{h}{2gr^2}}\left[(hs-s)^{\frac{1}{2}} - \frac{1}{2}h\text{versin}^{-1}\frac{2s}{h} + \tfrac{1}{2}\pi h\right] \quad . \; . \; . \; . \quad (143)$$

Making $s = r$ in (143), we find,

$$t = \sqrt{\frac{h}{2gr^2}}\left[(hr-r^2)^{\frac{1}{2}} - \frac{1}{2}h\text{versin}^{-1}\frac{2r}{h} + \frac{1}{2}\pi h\right] \quad . \; . \; . \; . \quad (144)$$

Which gives the time required for a body to fall from a distance h to the surface of the earth or sun.

Body Falling Under the Action of a Force that Varies as the Distance.

102. If the earth were homogeneous, a point within its surface would be attracted downward by a force that would be directly proportional to the distance from the centre (Calculus, p. 198). Let us then suppose the earth to be homogeneous and that an opening is made along a diameter from surface to surface. Denote the acceleration at the surface by g, the radius by r, and the acceleration at a

distance s from the centre by ϕ. From the principle stated, we have,

$$g : \phi :: r : s ; \quad \therefore \ \phi = \frac{g}{r} s \ . \ . \ . \ . \ . \ (145)$$

Substituting this in (92) and at the same time giving it the minus sign because it acts in the direction of s negative, we have,

$$\frac{d^2s}{dt^2} = -\frac{gs}{r}.$$

Multiplying by $2ds$ and integrating, we have,

$$\frac{ds^2}{dt^2} = -\frac{gs^2}{r} + C, \text{ or } v^2 = C - \frac{gs^2}{r} \ . \ . \ . \ . \ . \ (146)$$

Making $v = 0$ when $s = r$, we have $C = \frac{g}{r} r^2$, and this in (146) gives

$$v^2 = \frac{g}{r}(r^2 - s^2) \ . \ . \ . \ . \ . \ (147)$$

If we make $s = 0$, we find $v = \sqrt{gr}$, which is the velocity at the centre; it is also the maximum velocity. If the body moves on beyond the centre, s becomes negative, and we find that the velocities are equal at equal distances from the centre, whether s be positive or negative. When $s = -r$, v reduces to 0, and the body falls toward the centre again, and so on continually.

To find the time required for the body to fall from surface to surface, replace v^2 by its value, $\frac{ds^2}{dt^2}$ in (147) and solve with respect to dt, and making the result negative because t is a decreasing function of s, we have,

$$dt = -\sqrt{\frac{r}{g}} \frac{ds}{\sqrt{r^2 - s^2}} \ . \ . \ . \ . \ . \ (148)$$

Integrating (148) from r to $-r$ and denoting the corresponding time by t, we have,

$$t = \pi\sqrt{\frac{r}{g}} \quad . \quad . \quad . \quad . \quad . \quad (149)$$

which we shall see hereafter is the time of vibration of a simple pendulum whose length is r.

The kind of motion considered in this article is called **harmonic**, being the same as that of the particles of a vibrating cord, or spring.

When the body sets out from one extremity A' of the diameter $A'C'$ to fall through that diameter, suppose a second body to start from the same point and to travel around the semicircle $A'MC'$ with a constant velocity $\sqrt{gr}$, the maximum velocity of the first body. The time required for the second body to reach C' will be $\pi r \div \sqrt{gr}$ or $\pi\sqrt{\frac{r}{g}}$, that is, the two bodies will reach C' together.

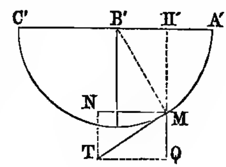

Fig. 97.

It is easily shown that the velocity of the first body is everywhere equal to the projection of the velocity of the second body. For let $B'H'$ be denoted by s; then will the velocity of the first body at H' be equal to $\sqrt{\frac{g}{r}(r^2 - s^2)}$. Draw $H'M$ perpendicular to $A'C'$, and at M draw MT to represent the velocity $\sqrt{gr}$, and let MN be its projection parallel to $A'C'$. We shall have $MN = MT \cos NMT$; but $\cos NMT$

$$= \cos H'MB' = \frac{\sqrt{r^2 - s^2}}{r}; \text{ hence, } MN = \sqrt{\frac{g}{r}(r^2 - s^2)},$$

that is, the velocity of the first point at H' is the projection of the corresponding velocity of the second point.

From what precedes, we may regard *harmonic motion* as the *projection of uniform circular motion*, that is, if a point revolves uniformly in a circle, the projection of the point on a diameter will move harmonically.

2°. Plane Curvilinear Motion.

Principle Employed.

103. If a material point moves in a plane curve, we may regard its path as resulting from two simultaneous motions, which are respectively parallel to two right lines lying in the plane of the curve. Thus, we may conceive the point to be moving parallel to the axis of X in accordance with a certain law, and parallel to the axis of Y in accordance with some other law; or we may conceive it to be moving in the direction of the tangent according to some law, and in the direction of the normal in accordance with some other law.

In accordance with Newton's second law we may study the motion of a body in any direction as though it had no motion in any other direction.

Motion of a Point Down a Curve in a Vertical Plane.

104. Let a body fall down a curve, situated in a vertical plane, under the action of gravity regarded as constant; and let the axis of Y be vertical, distances downward being positive.

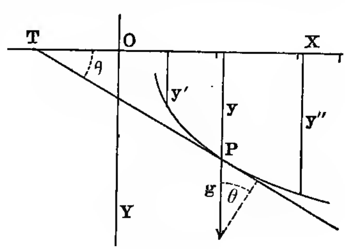

Fig. 98.

At any point of the curve whose ordinate is y, the acceleration due to gravity being denoted by g, we have for the tangential component of this acceleration $g \sin\theta$, or (Calc. p. 12) $g\dfrac{dy}{ds}$. Substituting this in (92), we have

$$\frac{d^2s}{dt^2} = g\frac{dy}{ds}, \text{ or } \frac{ds \cdot d^2s}{dt^2} = gdy.$$

Taking the integral between the limits y' and y we have, after reduction,

$$v^2 = 2g\ (y - y'), \text{ or } v = \sqrt{2g\ (y - y')} \quad . \quad . \quad . \quad . \quad . \quad (150)$$

The second member is the velocity due to the height $y-y'$. Hence, *the velocity generated in a body rolling down a curve, gravity being constant, is equal to that generated in falling freely through the same height.*

Replacing v in (150) by $\frac{ds}{dt}$ and solving, we have,

$$dt = \frac{ds}{\sqrt{2g\,(y-y')}} = \frac{ds}{dy}\,\frac{dy}{\sqrt{2g\,(y-y')}} \quad . \quad . \quad . \quad . \quad . \quad (151)$$

Integrating (151) between the proper limits, we have the time of falling through the height, $y - y'$.

Time of Descent on an Inverted Cycloid.

105. Let APB represent one half of a branch of an inverted cycloid, the origin being at A, and the values of y being positive downward. Its differential equation (Calc., p. 100) is

$$dx = \pm \frac{y\,dy}{\sqrt{2ry - y^2}} \quad . \quad . \quad . \quad . \quad (152)$$

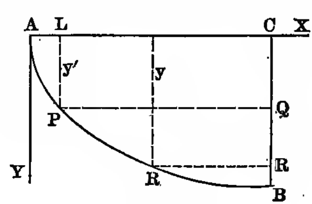

Fig. 99.

From (152) we find, by reduction,

$$\frac{ds}{dy} = \sqrt{\frac{2r}{2r - y}}.$$

Substituting in (151), and reducing, we have,

$$dt = \sqrt{\frac{r}{g}} \cdot \frac{dy}{\sqrt{(2r - y)(y - y')}} \quad . \; . \; . \; . \; . \quad (153)$$

To integrate the second member of (153), we make the distance $y - y' = z$; whence, $2r - y = 2r - y' - z$, and $dy = dz$. Substituting in (153), it becomes

$$dt = \sqrt{\frac{r}{g}} \cdot \frac{dz}{\sqrt{(2r - y')z - z^2}} \quad . \; . \; . \; . \; . \quad (154)$$

Applying formula [26], Calculus, we have,

$$t = \sqrt{\frac{r}{g}} \cdot \text{versin}^{-1} \frac{2z}{2r - y'} + C \quad . \; . \; . \; . \; . \quad (155)$$

If we suppose the body to fall from P to B, we have at P, $z = 0$, and at B we have, $z = 2r - y'$. Hence, the value of t between those limits is

$$t = \pi \sqrt{\frac{r}{g}}, \quad . \; . \; . \; . \; . \quad (156)$$

which is entirely independent of y'; that is, *the time required for a body to fall from any point of an inverted cycloid to its vertex is constant.*

Motion of Projectiles.

106. If a body be projected obliquely upward in a vacuum, and then abandoned to the force of gravity, it will be continually deflected from a rectilinear path, and, after describing a curvilinear path, called its **trajectory**, will finally reach the horizontal plane from which it started.

The starting-point is the **point of projection**; the distance from the point of projection to the point at which the

projectile again reaches the same horizontal plane is the **range**, and the time occupied is **the time of flight**. The only forces to be considered, are the *initial impulse* and *the force of gravity*. Hence, the trajectory will lie in a vertical plane through the direction of the initial impulse.

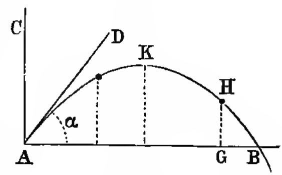

Fig. 100.

Let CAB be this plane, A the point of projection, AB the range, and AC a vertical through A. Take AB and AC as co-ordinate axes; denote the angle of projection, DAB, by α, and the velocity due to the initial impulse by v. Resolve v into two components, one in the direction AC, and the other in the direction AB. We have, for the former, $v \sin \alpha$, and, for the latter, $v \cos \alpha$.

The velocities, and, consequently, the spaces described in the direction of the co-ordinate axes, will (Art. 103) be entirely independent of each other. Denote the space described in the direction AC, in any time t, by y. The circumstances of motion in this direction, are those of a body projected vertically upward with an initial velocity, $v \sin \alpha$, and then continually acted on by the force of gravity. Hence, Equation (111) is applicable. Making, in that equation $h = y$, and $v' = v \sin \alpha$, we have,

$$y = v \sin \alpha \, t - \tfrac{1}{2} g t^2 \quad . \; . \; . \; . \; . \quad (157)$$

Denote the space described in the direction of the axis, AB, in the time t, by x. The only force in this direction is the component of the initial impulse. Hence, the motion will be uniform, and the first of equations (5) is applicable. Making $s = x$, and $v = v \cos \alpha$, we have,

$$x = v \cos \alpha \, t \quad . \; . \; . \; . \; . \quad (158)$$

If we suppose t to be the same in equations (157) and (158),

they will be simultaneous, and taken together, will make known the position of the projectile at any instant. Finding the value of t from (158), and substituting in (157), we have,

$$y = \frac{\sin\alpha}{\cos\alpha}x - \frac{gx^2}{2v^2\cos^2\alpha}, \quad . \quad . \quad . \quad . \quad . \quad (159)$$

which expresses the relation between x and y for all values of t; hence, it is the equation of the trajectory. Denoting the height due to v by h, we have, $v^2 = 2gh$, which reduces (159) to the form

$$y = \frac{\sin\alpha}{\cos\alpha}x - \frac{1}{4h\cos^2\alpha}x^2 \quad . \quad . \quad . \quad . \quad . \quad (160)$$

This equation satisfies the test $b^2 = 4ac$ (An. Geom., p. 158); hence, we conclude that the trajectory is a **parabola.**

To find the range, make $y = 0$ in (160); this gives,

$$x = 0, \quad \text{and} \quad x = 4h\sin\alpha\cos\alpha;$$

the *first* value of x corresponds to the point of projection, and the *second* is the value of the range. Denoting the range by r, and making $2\sin\alpha\cos\alpha = \sin 2\alpha$, we have,

$$r = 2h\sin 2\alpha \quad . \quad . \quad . \quad . \quad . \quad (161)$$

The range will be a maximum when $\alpha = 45°$, in which case we have,

$$r = 2h \quad . \quad . \quad . \quad . \quad . \quad (162)$$

If in (161) we replace α by $90° - \alpha$, the value of r will not be changed; hence, there are two angles of projection, complements of each other, that give the same range. The trajectories in the two cases are not the same, as may be shown by substituting the values of α, and $90° - \alpha$, in equation (160). The greater angle of projection gives a higher elevation, and consequently, the projectile descends in a line

nearer the vertical. It is for this reason that the gunner selects the *greater* of the two, when he desires to crush an object, and the *less* when he desires to batter, or overturn the object. If $\alpha = 90°$, the value of r is 0. That is, if a body be projected vertically upward, it will return to the point of projection.

To find the *time of flight,* make $x = r$ in (158), and deduce the value of t; this gives,

$$t = \frac{r}{v \cos \alpha} \quad . \quad . \quad . \quad . \quad . \quad (163)$$

If the range and initial velocity are constant, the time of flight will be greatest when α is greatest.

107. To find the highest point, K, we have simply to find the point where the tangent is horizontal. To do this, we find the differential coefficient of y from equation (160), and make it equal to 0; this gives,

$$\frac{dy}{dx} = \frac{\sin \alpha}{\cos \alpha} - \frac{x}{2h \cos^2 \alpha} = 0;$$

from which we find $x = 2h \sin \alpha \cos \alpha$, or half the range; substituting this in (160), we find $y = h \sin^2 \alpha$; these values of x and y are the co-ordinates of K.

To find the equation of the trajectory referred to parallel axes through K, we first change the sign of y in (160), so that ordinates may be positive downward; this gives,

$$-y = \frac{\sin \alpha}{\cos \alpha} x - \frac{1}{4h \cos^2 \alpha} x^2.$$

In this equation we make,

$$y = y' - h \sin^2 \alpha, \text{ and } x = x' + 2h \sin \alpha \cos \alpha,$$

and reduce: the resulting equation, after dropping the dashes, is,

$$x^2 = 4h \cos^2 \alpha \, y \quad . \quad . \quad . \quad . \quad . \quad (164)$$

In this parabola the parameter is $4h\cos^2\alpha$, and consequently the distance from K to the directrix is $h\cos^2\alpha$; adding this to the ordinate of K, which is $h\sin^2\alpha$, we see that the directrix of the curve is at a distance from AB equal to h.

The distance from K to the focus is equal to $h\cos^2\alpha$: hence, if $\alpha < 45°$, the focus is below AB; if $\alpha = 45°$, the focus is on AB; if $\alpha > 45°$, the focus is above AB.

108. To find an angle of projection such that the trajectory may pass through a given point, we substitute in (160), for the reciprocal of $\cos^2\alpha$, its value $1 + \tan^2\alpha$; that equation may then be written,

$$4hy = 4h\tan\alpha\, x - (1 + \tan^2\alpha)\, x^2.$$

Denoting the co-ordinates of the given point by x', and y', the equation of condition that the curve shall pass through it is,

$$4hy' = 4h\tan\alpha\, x' - (1 + \tan^2\alpha) x'^2,$$

which can be written,

$$\tan^2\alpha - \frac{4h}{x'}\tan\alpha = -\frac{4hy' + x'^2}{x'^2} \quad . \ . \ . \ . \ . \ (165)$$

Solving (165), we have,

$$\tan\alpha = \frac{2h \pm \sqrt{4h^2 - 4hy' - x'^2}}{x'} \quad . \ . \ . \ . \ . \ (166)$$

This shows that there are two angles of projection, under either of which, the point may be struck.

If we suppose,

$$x'^2 = 4h^2 - 4hy', \quad . \ . \ . \ . \ . \ (167)$$

the quantity under the radical sign will be 0, and the two angles of projection will become one.

If x' and y' be regarded as variables, equation (167) represents a parabola whose axis is a vertical line, through the

point of projection. Its vertex is at a distance, h, above the point, A, its focus is at A, and its parameter is $4h$, or twice the maximum range.

If we suppose,

$$x'^2 < 4h^2 - 4hy',$$

the point (x', y'), will lie within the parabola just described, the quantity under the radical sign will be positive, and there will be two real values of $\tan\alpha$, and, consequently, two angles of projection, under either of which the point may be struck.

If we suppose,

$$x'^2 > 4h^2 - 4hy',$$

the point (x', y'), will be without this parabola, the values of $\tan\alpha$ will both be imaginary, and there will be no angle under which the point can be struck.

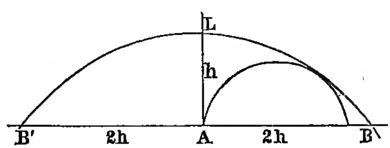

Fig. 101.

Let the parabola $B'LB$ represent the curve whose equation is

$$x'^2 = 4h^2 - 4hy'.$$

Conceive it to be revolved about AL, as an axis, generating a paraboloid of revolution. Then, from what precedes, we conclude, *first*, that every point within the surface may be reached from A, under two different angles of projection; *secondly*, that every point on the surface can be reached, but only by a single angle of projection; *thirdly*, that no point without the surface can be reached at all.

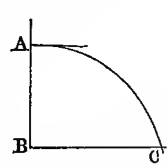

Fig. 102.

109. If a body is projected horizontally from an elevated point, A, its trajectory will be made known from equation (160) by

simply making $\alpha = 0$; whence, $\sin\alpha = 0$, and $\cos\alpha = 1$. Substituting and reducing, we have,

$$x^2 = -4hy \quad . \quad . \quad . \quad . \quad . \quad (168)$$

For every value of x, y is negative, which shows that the trajectory lies below the horizontal through the point of projection. If we suppose ordinates to be positive downward, we have,

$$x^2 = 4hy \quad . \quad . \quad . \quad . \quad . \quad (169)$$

To find the point at which the trajectory will reach any horizontal plane whose distance below A is h', we make $y = h'$ in (169), whence,

$$x = 2\sqrt{hh'} \quad . \quad . \quad . \quad . \quad . \quad (170)$$

On account of the resistance of the air, the results of the preceding discussion must be greatly modified. They approach more nearly to the observed phenomena, as the velocity is diminished and the density of the projectile increased. The atmospheric resistance increases as the square of the velocity, and as the cross section of the projectile exposed to the action of the resistance. In the air, it is found, under ordinary circumstances, that the maximum range is obtained by an angle of projection, not far from 34°.

NOTE.—In the following examples atmospheric resistance is neglected.

EXAMPLES.

1. What is the time of flight of a projectile in vacuum, when the angle of projection is 45°, and the range 6000 feet? *Ans.* 19.3 *sec.*

2. What is the range of a projectile, when the angle of projection is 30°, and the initial velocity 200 feet? *Ans.* 1076.9 *ft.*

3. The angle of projection under which a shell is thrown is 32°, and the range 3250 feet. What is the time of flight? *Ans.* 11.25 *sec.*, nearly.

4. Find the angle of projection and velocity of projection of a shell, so that its trajectory shall pass through two points, the co-ordinates of the first being $x = 1700$ *ft.*, $y = 10$ *ft.*, and of the second, $x = 1800$ *ft.*, $y = 10$ *ft.* *Ans.* $a = 39'19''$; $v = 2218.3$ *ft.*

5. At what elevation must a shell be projected with a velocity of 400 feet, that it may range 7500 feet on a plane which descends at an angle of 30°?

SOLUTION.—The co-ordinates of the point at which the shell strikes are,

$$x' = 7500 \cos 30° = 6495\,; \quad \text{and} \quad y' = -\,7500 \sin 30° = -\,3750.$$

And denoting the height due to the velocity 400 *ft.*, by h, we have,

$$h = \frac{v^2}{2g} = 2486 \textit{ ft.}$$

Substituting these values in the formula,

$$\tan a = \frac{2h \pm \sqrt{4h^2 - 4hy' - x'^2}}{x'},$$

and reducing, we have,

Ans. $a = 4° \; 34' \; 10''$; and $a = 55° \; 25' \; 41''$.

Tangential and Normal Components.

110. A point cannot move in a curve except under the action of an *incessant force,* whose direction is inclined to the direction of the motion. This force is called the **deflecting force,** and can be resolved into two components, one in the direction of the motion, and the other at right angles to it. The former acts simply to increase or diminish the velocity, and is called the **tangential force**: the latter acts to turn the point from its rectilinear direction, and being directed toward the centre of curvature is called the **centripetal force.**

The normal reaction, which is equal and directly opposed to the centripetal force, is called the **centrifugal force.**

To find expressions for the tangential and centripetal forces, let the acceleration due to the deflecting force at any

point, P, of the curve be resolved into components parallel to the axes OX and OY, and denote these components respectively by ϕ' and ϕ''. Let these components be again resolved into components acting tangentially along PT, and normally along PN. Denote the algebraic sum of the tangential components by T, and the algebraic sum of the normal components by N. Assuming the notation of the figure, we have,

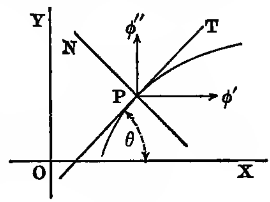

Fig. 103.

$$T = \phi' \cos\theta + \phi'' \sin\theta \quad . \quad . \quad . \quad . \quad . \quad (171)$$

$$N = \phi' \sin\theta - \phi'' \cos\theta \quad . \quad . \quad . \quad . \quad . \quad (172)$$

But from Art. 92,

$$\phi' = \frac{d^2x}{dt^2}, \quad \text{and} \quad \phi'' = \frac{d^2y}{dt^2};$$

and from Calculus, p. 12, we have,

$$\cos\theta = \frac{dx}{ds}, \quad \text{and} \quad \sin\theta = \frac{dy}{ds}.$$

Substituting these in (171), and reducing, on the supposition that t is the independent variable, we have,

$$T = \frac{d^2x}{dt^2} \cdot \frac{dx}{ds} + \frac{d^2y}{dt^2} \cdot \frac{dy}{ds}$$

$$= \frac{d\,(dx^2 + dy^2)}{dt^2 \cdot 2ds} = \frac{d\,(ds^2)}{dt^2 \cdot 2ds} = \frac{d^2s}{dt^2} \quad . \quad . \quad . \quad . \quad . \quad (173)$$

Substituting the same quantities in (172), we have,

$$N = \frac{d^2x \cdot dy - d^2y \cdot dx}{dt^2 \cdot ds} = -\frac{ds^2}{dt^2} \cdot \frac{dx \cdot d^2y - dy \cdot d^2x}{ds^3}.$$

But the second factor of the last member is equal to the reciprocal of the radius of curvature denoted by R (Calculus, p. 65); substituting this in the preceding equation, and reducing, we have,

$$N = -\frac{v^2}{R}, \quad . \quad . \quad . \quad . \quad . \quad (174)$$

which is equal and directly opposed to the acceleration due to the centrifugal force. Denoting the centrifugal force, when the mass of the body is M, by F, we have,

$$F = \frac{Mv^2}{R} \quad . \quad . \quad . \quad . \quad . \quad (175)$$

Hence, *the centrifugal force of a body, whose mass is M, that is, the force that it exerts normally to the curve which it is compelled to describe, varies directly as its mass into the square of its velocity, and inversely as the radius of curvature of the curve.*

The subject of centrifugal force will be further considered hereafter.

3°. Periodic Motion.

Rectilinear and Curvilinear Vibration.

111. Periodic motion is a kind of variable motion, in which the spaces described in certain equal periods of time are equal. This kind of motion is exemplified in the phenomena of **vibration**, of which there are two cases.

1st. *Rectilinear vibration.* Theory indicates, and experiment confirms the fact, that if a particle of an elastic fluid be slightly disturbed from its place of rest, and then abandoned, it will be urged back by a force, varying *directly* as its distance from the position of equilibrium; on reaching this position, the particle will, by virtue of its inertia, pass to the other side, again to be urged back, and so on.

To find the time required for the particle to pass from one extreme position to the opposite one and back, let us denote the displacement at any time t by s, and the acceleration due to the *restoring force* by ϕ; then, from the law of the force, we shall have $\phi = n^2 s$, in which n is constant for the same fluid at the same temperature. Substituting for ϕ its value, Equation (92), and recollecting that ϕ acts in a direction contrary to that in which s is estimated, we have,

$$-\frac{d^2 s}{dt^2} = n^2 s \quad . \quad . \quad . \quad . \quad . \quad (176)$$

Multiplying both members of (176) by $2ds$, we have,

$$-\frac{2ds\, d^2 s}{dt^2} = 2n^2 s ds\,;$$

whence, by integration,

$$-\frac{ds^2}{dt^2} = n^2 s^2 + C = -v^2.$$

The velocity v will be 0 when s is greatest possible; denoting this value of s by a, we shall have,

$$n^2 a^2 + C = 0\,; \quad \text{whence,} \quad C = -n^2 a^2.$$

Substituting this value of C in the preceding equation, it becomes,

$$v^2 = \frac{ds^2}{dt^2} = n^2 (a^2 - s^2)\,; \quad \text{whence,} \quad ndt = \frac{ds}{\sqrt{a^2 - s^2}} \quad . \quad . \quad . \quad (177)$$

Integrating the last equation, we have,

$$nt + C = \sin^{-1}\frac{s}{a} \quad . \quad . \quad . \quad . \quad . \quad (178)$$

Taking the integral between the limits $s = +a$ and

$s = -a$, and denoting the corresponding time by $\frac{1}{2}\tau$, τ being the time of a double vibration, we have,

$$\tfrac{1}{2}n\tau = \pi\,; \quad \text{whence,} \quad \tau = \frac{2\pi}{n}.$$

The value of τ is independent of the extent of the excursion, and dependent only upon n. Hence, in the same medium, and at the same temperature, the time of vibration is *constant.*

2ndly. *Curvilinear vibration.* Let ABC be a vertical plane curve, symmetrical with respect to DB. Let AC be a horizontal line, and denote the distance EB by h. If a body were placed at A and abandoned to the action of its own weight, being constrained to remain on the curve, it would, in accordance with preceding principles, move toward B with an accelerated motion, and, on arriving at B, would possess a velocity due to the height h. By virtue of its inertia it would ascend the branch BC with retarded motion, and would finally reach C, where its velocity would be 0. The body would then be in the same condition that it was at A, and would, consequently, descend to B and again ascend to A, whence it would again descend, and so on. Were there no retarding causes, the motion would continue for ever. From what has preceded, it is obvious that the time occupied by the body in passing from A to B is equal to that in passing from B to C, and also the time in passing from C to B is equal to that in passing from B to A. Further, the velocities of the body when at G and H, any two points lying on the same horizontal, are equal, either being that due to the height EK.

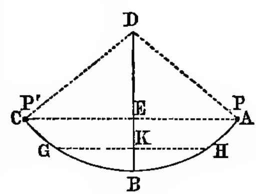

Fig. 104.

Angular Velocity.—Angular Acceleration.

112. When a body revolves about an axis, its points, being at different distances from the axis, will have different velocities. The **angular velocity** is the velocity of a point whose distance from the axis is equal to 1. To obtain the velocity of any other point, we multiply its distance from the axis by the angular velocity. To find a general expression for the velocity of any point of a revolving body, let us denote the angular velocity by ω, the space passed over by a point at the unit's distance from the axis in the time dt, by $d\theta$. The quantity $d\theta$ is an infinitely small arc, having a radius equal to 1; and, as in Art. 92, it is plain that we may regard the angular motion as uniform, during the infinitely small time dt. Hence, as in Article 92, we have,

$$\omega = \frac{d\theta}{dt}.$$

If we denote the distance of any point from the axis by l, and its velocity by v, we shall have,

$$v = l\omega; \text{ or, } v = l\frac{d\theta}{dt} \quad . \quad . \quad . \quad . \quad . \quad (179)$$

The **angular acceleration** due to a force is the rate at which the force can impart angular velocity. If we denote it by ϕ_1, we have, as in Art. 92,

$$\phi_1 = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}.$$

The corresponding acceleration of a point whose distance from the axis is l, will be,

$$l\phi_1 = l\frac{d^2\theta}{dt^2} \quad . \quad . \quad . \quad . \quad . \quad (180)$$

The corresponding measure of the moving force is found by multiplying (180) by the mass m.

The Simple Pendulum.

113. A **pendulum** is a heavy body suspended from a horizontal axis, about which it is free to vibrate. In order to investigate the circumstances of vibration, let us first consider the hypothetical case of a single material point vibrating about an axis, to which it is attached by a rod destitute of weight. Such a pendulum is called a **simple pendulum.** The laws of vibration, in this case, will be identical with those explained in Art. 111, the arc ABC being the arc of a circle. The motion is, therefore, *periodic.*

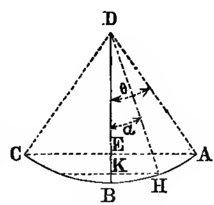

Fig. 105.

Let ABC be the arc through which the vibration takes place, and denote its radius by l. The angle CDA is called the **amplitude** of vibration; half of this angle ADB, denoted by α, is called the **angle of deviation**; and l is called **the length of the pendulum.** If the point starts from rest, at A, it will, on reaching any point H, of its path, have a velocity v, due to the height EK, denoted by h. Hence,

$$v = \sqrt{2gh} \quad . \; . \; . \; . \; . \quad (181)$$

If we denote the variable angle HDB by θ, we shall have $DK = l \cos\theta$; we shall also have $DE = l\cos\alpha$; and since h is equal to $DK - DE$, we shall have,

$$h = l(\cos\theta - \cos\alpha).$$

Which, being substituted in the preceding formula, gives,

$$v = \sqrt{2gl(\cos\theta - \cos\alpha)}.$$

From the preceding article, we have,

$$v = l\frac{d\theta}{dt}.$$

Equating these two values of v, we have,

$$l\frac{d\theta}{dt} = \sqrt{2gl(\cos\theta - \cos\alpha)}.$$

Whence, by solving with respect to dt,

$$dt = \sqrt{\frac{l}{2g}} \cdot \frac{d\theta}{\sqrt{\cos\theta - \cos\alpha}} \quad . \quad . \quad . \quad . \quad . \quad (182)$$

If we develop $\cos\theta$ and $\cos\alpha$ into series, by McLaurin's theorem, we shall have,

$$\cos\theta = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{1.2.3.4} - \text{etc.};$$

$$\cos\alpha = 1 - \frac{\alpha^2}{2} + \frac{\alpha^4}{2.3.4} - \text{etc.}$$

When α is very small, say two or three degrees, θ being still smaller, we may neglect all the terms after the second as inappreciable, giving

$$\cos\theta - \cos\alpha = \tfrac{1}{2}(\alpha^2 - \theta^2).$$

Substituting in equation (182), it becomes,

$$dt = \sqrt{\frac{l}{g}} \cdot \frac{d\theta}{\sqrt{\alpha^2 - \theta^2}} \quad . \quad . \quad . \quad . \quad . \quad (183)$$

Integrating equation (183), we have,

$$t = \sqrt{\frac{l}{g}}\sin^{-1}\frac{\theta}{\alpha} + C.$$

Taking the integral between the limits $\theta = \alpha$ and $\theta = -\alpha$, t will denote the time of one vibration, and we shall have,

$$t = \pi\sqrt{\frac{l}{g}} \quad . \quad . \quad . \quad . \quad . \quad (184)$$

Hence, *the time of vibration of a simple pendulum is equal to the number* 3.1416, *multiplied into the square root of the quotient obtained by dividing the length of the pendulum by the force of gravity.*

For a pendulum, whose length is l', we shall have,

$$t' = \pi \sqrt{\frac{l'}{g}} \quad . \quad . \quad . \quad . \quad . \quad (185)$$

From equations (184) and (185), we have, by division,

$$\frac{t}{t'} = \frac{\sqrt{l}}{\sqrt{l'}}; \quad \text{or,} \quad t : t' :: \sqrt{l} : \sqrt{l'} \quad . \quad . \quad . \quad . \quad . \quad (186)$$

That is, *the times of vibration of two simple pendulums, are to each other as the square roots of their lengths.*

If we suppose the lengths of two pendulums to be the same, but the force of gravity to vary, as it does slightly in different latitudes, and at different elevations, we shall have,

$$t = \pi \sqrt{\frac{l}{g}}, \quad \text{and} \quad t'' = \pi \sqrt{\frac{l}{g''}}.$$

Whence, by division,

$$\frac{t}{t''} = \sqrt{\frac{g''}{g}}, \quad \text{or,} \quad t : t'' :: \sqrt{g''} : \sqrt{g} \quad . \quad . \quad . \quad . \quad . \quad (187)$$

That is, *the times of vibration of the same simple pendulum, at two different places, are to each other inversely as the square roots of the forces of gravity at the two places.*

If we suppose the times of vibration to be the same, and the force of gravity to vary, the lengths will vary also, and we shall have,

$$t = \pi \sqrt{\frac{l}{g}}, \quad \text{and} \quad t = \pi \sqrt{\frac{l'}{g'}}.$$

Equating these values and squaring, we have,

$$\frac{l}{g}=\frac{l'}{g'}; \quad \text{or,} \quad l : l' :: g : g' \; . \; . \; . \; . \; . \quad (188)$$

That is, *the lengths of simple pendulums which vibrate in equal times at different places, are to each other as the forces of gravity at those places.*

Vibrations of equal duration are called **isochronal.**

De l'Ambert's Principle.

114. When several bodies are rigidly connected, it often happens that they are constrained to move in a different manner from what they would, if free. Some move *faster* and some *slower* than they would, were it not for the connection. In the former case there is a *gain,* and in the latter a *loss,* of moving force, in consequence of the connection. It is obvious that the resultant of all the impressed forces is equal to that of all the effective forces, for if the latter were reversed, they would hold the former in equilibrio.

Hence, *all the moving forces lost and gained in consequence of the connection are in equilibrium.*

This is known as **De l'Ambert's principle.**

The Compound Pendulum.

115. A **compound pendulum** is a body free to vibrate about a horizontal axis, called the **axis of suspension.** The straight line drawn from the centre of gravity of the pendulum perpendicular to the axis of suspension is called the **axis of the pendulum.**

In practical applications, the pendulum is so shaped that the plane through the axis of suspension and the centre of gravity divides it symmetrically.

Were the particles of the pendulum entirely disconnected,

but constrained to remain at invariable distances from the axis of suspension, we should have a collection of simple pendulums. Those at equal distances from the axis would vibrate in equal times, and those unequally distant would vibrate in unequal times. The particles nearest the axis would vibrate more rapidly than the compound pendulum, and those most remote would vibrate slower; hence, there must be intermediate points that would vibrate in the same time as the pendulum. These points lie on the surface of a circular cylinder whose axis is that of suspension; the point in which this cylinder cuts the axis of the pendulum is called the **centre of oscillation.** If the entire mass of the pendulum were concentrated at this point, the time of its vibration would be unchanged.

Hence, **the centre of oscillation of a compound pendulum** is a point of its axis, at which, if the mass of the pendulum were concentrated, its time of vibration would be unchanged. A line drawn through this point, parallel to the axis of suspension, is called the **axis of oscillation.** The distance between the axis of oscillation and the axis of suspension is the length of an **equivalent simple pendulum,** that is, of a simple pendulum, whose time of vibration is the same as that of the compound pendulum.

Angular Acceleration of a Compound Pendulum.

116. Let CK be a compound pendulum, C its axis of suspension, G its centre of gravity, and suppose the plane of the paper to pass through the centre of gravity, G, and perpendicular to the axis, C. We may regard the pendulum as made up of infinitely small filaments, parallel to the axis of suspension, and consequently perpendicular to the paper. The circumstances of vibration will be unchanged if we suppose each element to be concentrated in the point where it meets the plane of the paper. Denote the mass of any such

element, as S, by m, its distance from C, by r, and the mass of the entire pendulum by M.

Through C draw a horizontal line, CB, and draw SH, GA, and OB, perpendicular to it. On HS prolonged, take SE to represent the moving force *impressed* on S. Then will SE be equal to mg, (Eq. 93), and its moment with respect to C will be $mg \times HC$. Denote the *angular acceleration* by ϕ_1; then will the actual acceleration of S, in a direction perpendicular to SC, be equal to $r\phi_1$ (Eq. 180), and the *effective* moving force to $mr\phi_1$; because this force acts at right angles to SC, its moment is equal to $mr^2\phi_1$. Because mg is the moving force *impressed* on S, and $mr\phi_1$ the *effective* moving force, the expression, $mg - mr\phi_1$, will be the moving force lost or gained by S in consequence of its connection with the other particles. There will be a loss when mg is *greater* than $mr\phi_1$, and a gain when mg is less than $mr\phi_1$. The moment of this force with respect to C is equal to $mg \times CH - mr^2\phi_1$. Similar expressions may be found for each of the elementary particles of the pendulum.

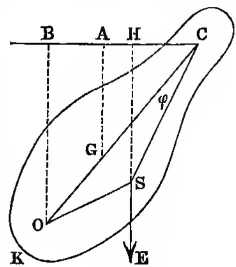

Fig. 106.

By de l'Ambert's principle, the moving forces lost and gained, in consequence of the connection of the parts, are in equilibrium; hence, the algebraic sum of their moments with respect to the axis, C, is equal to 0, that is,

$$\Sigma\,(mg \times CH) - \Sigma\,(mr^2\phi_1) = 0.$$

But ϕ_1 and g are the same for each particle; hence,

$$\phi_1 = \frac{\Sigma\,(m \times CH)}{\Sigma\,(mr^2)}g.$$

From the principle of moments, we have,

$$\Sigma\,(m \times CH) = M \times CA.$$

Substituting above, we have, finally,

$$\phi_1 = \frac{M \times CA}{\Sigma\,(mr^2)} g \quad . \quad . \quad . \quad . \quad . \quad (189)$$

That is, *the angular acceleration varies as, CA, the lever arm of the weight of the pendulum.*

The expression $\Sigma\,(mr^2)$ is called the **moment of inertia** of the body with respect to the axis of suspension, Mg is the weight of the body, and $Mg \times CA$ is the moment of the weight, with respect to the same axis.

Hence, *the angular acceleration is equal to the moment of the weight, divided by the moment of inertia, both taken with respect to the axis of suspension.*

Length of an Equivalent Simple Pendulum.

117. To find the length of a simple pendulum that will vibrate in the same time as the given compound pendulum, let O be the centre of oscillation, and draw OB perpendicular to CB. Denote CO by l, and CG by k. Were the entire mass concentrated at O, each value of mr would become equal to ml, and we should have, for its moment of inertia, Ml^2, and for the moment of the mass, $M \times CB$, and for the angular acceleration,

$$\phi_1 = \frac{M \times CB}{Ml^2} g.$$

But the pendulum is to vibrate in the same time, whether it exist as a compound pendulum, or as a simple pendulum, its mass being concentrated at its centre of oscillation; the value of ϕ_1 must, therefore, be the same in both cases. Placing the value just deduced equal to that in equation (189), we have,

$$\frac{M \times CB}{Ml^2} g = \frac{M \times CA}{\Sigma\,(mr^2)} g;$$

whence, by reduction,

$$Ml^2 = \Sigma(mr^2) \times \frac{CB}{CA}.$$

From the similar triangles, CGA and COB, we have,

$$\frac{CB}{CA} = \frac{l}{k}.$$

Substituting, and reducing, we have,

$$l = \frac{\Sigma(mr^2)}{Mk} \quad . \quad . \quad . \quad . \quad . \quad (190)$$

That is, *the length of the equivalent simple pendulum is equal to moment of inertia of the pendulum divided by the moment of its mass, both taken with respect to the axis of suspension.*

Reciprocity of Axes of Suspension and Oscillation.

118. Let C be the axis of suspension, O the centre of oscillation, and let a line be drawn through O parallel to the axis of suspension. This line is called **the axis of oscillation.** Let the plane of the paper be taken as before, and suppose the elements projected on it, as in Article 116.

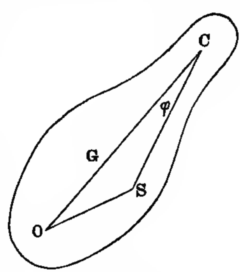

Fig. 107.

Let S be any element, and denote its distance from the axis of suspension by r, and from the axis of oscillation by t; denote OC by l, and the angle OCS by ϕ.

If the axis of oscillation be taken as an axis of suspension, and the length of the corresponding simple pendulum, denoted by l', we have, from the preceding article,

$$l' = \frac{\Sigma(mt^2)}{M(l-k)} \quad . \quad . \quad . \quad . \quad . \quad (191)$$

In the triangle, OSC, we have,

$$t^2 = r^2 + l^2 - 2rl\cos\phi\,;$$

hence,

$$\Sigma(mt^2) = \Sigma(mr^2) + \Sigma(ml^2) - 2\Sigma(mr\cos\phi)l.$$

But, from equation (190), we have,

$$\Sigma(mr^2) = Mkl\,;$$

and because l is invariable, we have,

$$\Sigma(ml^2) = \Sigma(m)l^2 = Ml^2\,;$$

if we suppose CO horizontal, $r\cos\phi$, the projection of r on CO, will be the lever-arm of m, and the expression, $\Sigma\,(mr\cos\phi)$, will be the algebraic sum of the moments of the elementary masses with respect to C; hence, we shall have,

$$\Sigma\,(mr\cos\phi)l = Mkl.$$

Substituting for these expressions their values given above, and putting the value of $\Sigma\,(mt^2)$, thus found, in (191), we have,

$$l' = \frac{Mkl + Ml^2 - 2Mkl}{M(l-k)} = \frac{M(l^2 - kl)}{M(l-k)}\,;$$

or,

$$l' = l \quad . \; . \; . \; . \; . \quad (192)$$

Hence, *the axis of suspension and oscillation are convertible;* that is, *if either be taken as an axis of suspension, the other will be the axis of oscillation.*

This property of the compound pendulum is employed to determine the length of the seconds' pendulum, and the value of the force of gravity at different places on the surface of the earth.

The Reversible Pendulum.

119. A reversible pendulum sufficiently accurate for purposes of illustration may be constructed as follows. A rectangular bar of steel, CD, about four feet long, is provided with two knife-edge axes, A and B, having their edges turned toward each other; these axes are attached to sliding sleeves with clamp screws, so that they may be set at different points of the bar CD; the axes should be mounted so that they shall be perpendicular to the plane of the bar, and so that their plane shall pass through the axis of the bar.

Fig. 108.

The pendulum thus constructed is suspended on horizontal plates of hard steel, so placed that the pendulum may vibrate freely between them, and around either axis at pleasure.

To adjust the axes A and B so the times of oscillation around them shall be equal, we proceed by the method of approximation. Set the axis A in some convenient position, and allow the pendulum to vibrate; note the time required to make, say 100 vibrations; then, dividing this by 100, we have the time of a single vibration. Substitute this value for t, in (184), and deduce the corresponding value of l. Set the axis B at a distance from A equal to l, and we have a first approximation.

Reversing the pendulum, and repeating the operation, we find a second approximation. By continuing this process of approximation, we ultimately find two positions of the axes around which the times of vibration are equal. The distance between the axes is then the length of the equivalent simple pendulum.

Knowing the length of the equivalent simple pendulum and the corresponding time of vibration, we may determine the acceleration due to the force of gravity from formula (184);

we may also find the length of a **seconds' pendulum**; that is, of a pendulum that vibrates once in a second, from the proportion (186), by making $t' = 1$, and substituting the known values of t and l.

The pendulum used for scientific purposes is far more complex in its modes of adjustment and use than the one above described, and far greater precautions are taken to avoid errors. Pendulums of this kind have been used for determining the figure of the earth, and for various other scientific purposes.

By a series of carefully conducted experiments, it has been found that the length of a seconds' pendulum in the Tower of London is 3.2616 ft., or 39.13921 inches.

Experiments made in different latitudes show that the force of gravity continually increases from the equator toward either pole. According to Plantamour's modification of Bessel's barometric formula, the value of g is given for any latitude by the formula

$$g = g'(1 - .0026257 \cos 2L), \quad . \quad . \quad . \quad . \quad . \quad (193)$$

in which g' is its value in latitude 45°, and L is the latitude of the given place.

According to Airy, the value of g' is about 32.17 *ft.* This, in (193), gives

$$g = 32.17(1 - .0026257 \cos 2L).$$

For the latitude of New York (40° 45′), we have,

$$g = 32.16 \, ft. \text{ nearly.}$$

Formula (193) takes account of the variation of g, due to latitude only. It may be still further modified to take account of variation in height above the level of the sea, as follows: let g be the value of gravity at the level of the sea in

the latitude L, and let g'' be its value in the same latitude at an elevation above the sea level denoted by z; if we denote the radius of the earth by R, we shall then have, by the Newtonian law,

$$g : g'' :: (R + z)^2 : R^2, \quad \text{or,} \quad g'' = g\frac{R^2}{(R + z)^2}.$$

Developing the denominator, and dividing R^2 by it, neglecting all terms of the quotient after the second, as insignificant, we have,

$$g'' = g\left(1 - \frac{2z}{R}\right).$$

Substituting in (193), we have,

$$g'' = g'(1 - .0026257 \cos 2L)\left(1 - \frac{2z}{R}\right) \quad . \quad . \quad . \quad . \quad . \quad (194)$$

From which we may find the value of gravity in any latitude, and at any elevation. If z is given in feet, R must also be given in feet: it is assumed that,

$$R = 20{,}886{,}860 \, ft.$$

Practical Application of the Pendulum.

120. One of the most important uses of the pendulum is to regulate the motion of clocks. A **clock** consists of a train of wheelwork, the last wheel of the train connecting with a pendulum-rod by a piece of mechanism called an **escapement.** The wheelwork is kept in motion by a descending weight, or by the elastic force of a spring, and the wheels are so arranged that one tooth of the last wheel in the train escapes from the pendulum-rod at each vibration of the pendulum, or at each **beat.** The number of beats is rendered visible on a dial-plate by indices, called **hands.**

On account of expansion and contraction, the length of the pendulum is liable to variation, which gives rise to irregularity in the times of vibration. To obviate this, and to render the times of vibration uniform, several devices have been resorted to, giving rise to what are called **compensating pendulums.** We shall indicate two of the most important of these, observing that the remaining ones are nearly the same in principle, differing only in mode of application.

Graham's Mercurial Pendulum.

121. Graham's mercurial pendulum consists of a rod of steel about 42 inches long, branched toward its lower end, to embrace a cylindrical glass vessel 7 or 8 inches deep, and having between 6 and 7 inches of this depth filled with mercury. The exact quantity of mercury, being dependent on the weight and expansibility of the other parts of the pendulum, may be determined by experiment in each case. When the temperature increases, the steel rod is lengthened, and, at the same time, the mercury rises in the cylinder. When the temperature decreases, the steel bar is shortened, and the mercury falls in the cylinder. By a proper adjustment of the quantity of mercury, the effect of the lengthening or shortening of the rod is exactly counterbalanced by the rising or falling of the centre of gravity of the mercury, and the axis of oscillation is kept at an invariable distance from the axis of suspension.

Harrison's Gridiron Pendulum.

122. Harrison's gridiron pendulum consists of five rods of steel and four of brass, placed alternately with each other, the middle rod, or that from which the **bob** is suspended, being of steel. These rods are connected by cross-pieces in such a manner that, whilst the expansion of the steel rods tends to elongate the pendulum, or lower the

bob, the expansion of the brass rods tends to shorten the pendulum, or raise the bob. By duly proportioning the sizes and lengths of the bars, the axis of oscillation may be maintained at an invariable distance from the axis of suspension. From what has preceded, it follows that whenever the distance from the axis of oscillation to the axis of suspension remains invariable, the times of vibration must be absolutely equal at the same place. The pendulums just described are principally used for astronomical clocks, where great accuracy and uniformity in the measure of time are indispensable.

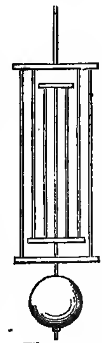

Fig. 109.

Basis of a System of Weights and Measures.

123. The pendulum is of further importance, in a practical point of view, in furnishing the standard that has been made the basis of the English system of weights and measures.

It was enacted by Parliament, in 1824, that the distance between the centres of two gold studs in a certain described brass bar, the bar being at a temperature of 62° F., should be an "**imperial standard yard.**" To be able to restore it in case of its destruction, it was enacted that the yard should be considered as bearing to the length of the seconds' pendulum in the latitude of London, in vacuum, and at the level of the sea, the ratio of 36 to 39.1393. From the yard, every other unit of linear measure may be derived, and thence all measures of area and volume.

It was also enacted that a certain described brass weight, made in 1758, and called 2 *lbs.* Troy, should be regarded as authentic, and that a weight equal to one half that should be "**the imperial standard Troy pound.**" The $\frac{1}{5760}$th part of the **Troy pound** was called a **grain**, of which 7000 constituted a **pound avoirdupois.** To provide for the con-

tingency of a loss of the standard, it was connected with the system of measures, by enacting, that if lost, it should be restored by allowing 252.724 grains for the weight of a cubic inch of distilled water, at 62° F., the water being weighed in vacuum and by brass weights. From the grain thus established, all other units of weight may be derived.

Our own system of weights and measures is the same as that of the English.

EXAMPLES.

1. The length of a seconds' pendulum is 39.13921 *in.* If it be shortened 0.130464 *in.*, how many vibrations will be gained in a day of 24 hours?
Ans. 144 *nearly.*

2. A seconds' pendulum on being carried to the top of a mountain, was observed to lose 5 vibrations per day of 86400 seconds. Required the height of the mountain, reckoning the radius of the earth at 4000 miles. *Ans.* $h = 0.2315$ *mi.* $= 1222$ *ft.*

3. What is the time of vibration of a pendulum whose length is 60 *in.*, when the force of gravity is $32\frac{1}{6}$ *ft.*? *Ans.* 1.2387 *sec.*

4. How many vibrations will a pendulum 36 inches in length make in one minute, the force of gravity being the same as before?
Ans. 62.53.

5. A pendulum makes 43170 vibrations in 12 hours. How much must it be shortened that it may beat seconds? *Ans.* .0544 *in.*

6. In a certain latitude, the length of a pendulum vibrating seconds is 39 inches. What is the length of a pendulum vibrating seconds, in the same latitude, at the height of 21000 feet above the first station, the radius of the earth being 3960 miles? *Ans.* 38.9218 *in.*

7. If a pendulum make 40000 vibrations in 6 hours, at the level of the sea, how many vibrations will it make in the same time, at an elevation of 10560 feet, the radius of the earth being 3960 miles?
Ans. 39979.8.

8. What is the length of a pendulum that will beat sidereal seconds, the length of the sidereal day being 23 *hrs.* 56 *min.* 4 *sec.*?
Ans. 39.0324 inches.

9. What is the length of a pendulum that makes as many vibrations per minute as it is inches long? *Ans.* 52.03 inches.

VI.—CENTRIFUGAL FORCE.—MOMENT OF INERTIA.

Centrifugal Force in Terms of Angular Velocity.

124. When a material point is constrained to move in a curve it offers a resistance to the force that deflects it from its rectilineal path. This resistance, as stated in Article 110, is called the **centrifugal force.** It is shown, in the article referred to, that the value of the centrifugal force is given by the Equation,

$$F = m\frac{v^2}{r}, \quad . \quad . \quad . \quad . \quad . \quad (195)$$

in which m is the *mass* of the particle, v its *lineal velocity,* and r the *radius of curvature* of its path at the instant in question.

If we denote the angular velocity of the point around the centre of curvature by ω, its linear velocity will be equal to $r\omega$, and this substituted in (195), gives

$$F = mr\omega^2 \quad . \quad . \quad . \quad . \quad . \quad (196)$$

This equation is often more convenient than (195). If we suppose the point to be restrained by a rigid curve, the centrifugal force is equal, and directly opposed to the *reaction* of the curve. If the material point m is whirled around a fixed point, being retained by a string, the centrifugal force is the measure of the **tension of the string.**

Centrifugal Force of an Extended Mass.

125. We have supposed, in what precedes, that the dimensions of the body under consideration are extremely small; let us next examine the case of a body, of any dimensions whatever, constrained to revolve about a fixed axis. If the body be divided into infinitely small elements, whose directions are parallel to the axis, the centrifugal force of each element will be equal to the mass of the element *into* the square of its velocity, divided by its distance from the axis. If a plane be passed through the centre of gravity of the body, perpendicular to the axis, we may, without impairing the generality of the result, suppose the mass of each element concentrated at the point in which this plane cuts the line of direction of the element.

Let XCY be the plane through the centre of gravity perpendicular to the axis of revolution, AB the projection of the body on the plane, and C the point in which it cuts the axis. Take C as the origin of a system of rectangular co-ordinates; let CX be the axis of X, CY the axis of Y, and m be the point at which the mass of one filament is concentrated, and denote that mass by m. Denote the co-ordinates of m by x and y, its distance from C by r, and its velocity by v. The centrifugal force of the mass, m (Eq. 196), is equal to

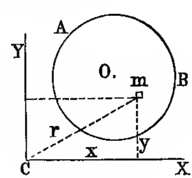

Fig. 110.

$$mr\,\omega^2.$$

Let this force be resolved into components parallel to CX and CY. We have, for these components,

$$mr\,\omega^2 \cos m\,CX, \quad \text{and} \quad mr\,\omega^2 \sin m\,CX.$$

But, from the figure,

$$\cos m\,CX = \frac{x}{r}, \quad \text{and} \quad \sin m\,CX = \frac{y}{r}.$$

Substituting these in the preceding expressions, and reducing, we have, for the components,

$$mx\,\omega^2, \quad \text{and} \quad my\,\omega^2.$$

Similar expressions may be deduced for each of the other filaments. If we denote the resultant of the components parallel to CX by X, and of those parallel to CY by Y, we have,

$$X = \Sigma\,(mx)\,\omega^2, \quad \text{and} \quad Y = \Sigma\,(my)\,\omega^2.$$

If we denote the mass of the body by M, and suppose it concentrated at its centre of gravity, O, whose co-ordinates are x_1, and y_1, and whose distance from C is r_1, we shall have, from the principle of the centre of gravity (Art. 37),

$$\Sigma\,(mx) = Mx_1, \quad \text{and} \quad \Sigma\,(my) = My_1.$$

Substituting above, we have,

$$X = M\omega^2 x_1, \quad \text{and} \quad Y = M\omega^2 y_1.$$

If we denote the resultant force by R, we have,

$$R = \sqrt{X^2 + Y^2} = M\omega^2\sqrt{x_1^2 + y_1^2} = M\omega^2 r_1 \quad . \quad . \quad . \quad . \quad . \quad (197)$$

The direction of the resultant R, Eq. (9), is given by the equations,

$$\cos a = \frac{X}{R} = \frac{x_1}{r_1}; \qquad \sin a = \frac{Y}{R} = \frac{y_1}{r_1}; \quad . \quad . \quad . \quad . \quad . \quad . \quad (198)$$

that is, the resultant passes through C.

But equations (197) and (198) are expressions for the intensity of the centrifugal force of the mass M concentrated at O.

Hence, *the centrifugal force of an extended mass, constrained to revolve about a fixed axis, is the same as though the mass were concentrated at its centre of gravity.*

Experimental Illustrations.

126. The principles relating to centrifugal force admit of experimental illustration. The instrument represented in the figure may be employed to show the value of the centrifugal force. A is a vertical axle, on which is mounted a wheel, F, communicating with a train of wheelwork, by means of which the axle may be made to revolve with any angular velocity. At the upper end of the axle is a forked branch, BC, sustaining a stretched wire. D and E are balls pierced by the wire, and free to move along it. Between B and E is a spiral spring, whose axis coincides with the wire.

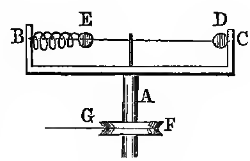

Fig. 111.

Immediately below the spring, on the horizontal part of the fork, is a scale for determining the distance of the ball, E, from the axis, and for measuring the degree of compression of the spring. Before using the instrument, the force required to produce any degree of compression of the spring is determined experimentally, and marked on the scale.

If a motion of rotation be communicated to the axis, the ball D will at once recede to C, but the ball E will be restrained by the spring. As the velocity of rotation increases, the spring is compressed more and more, and the ball E approaches B. By a suitable arrangement of wheelwork, the

angular velocity of the axis corresponding to any compression may be ascertained. We have, therefore, all the data necessary to verify the law of centrifugal force.

If a circular hoop of flexible material be mounted on one of its diameters, its lower point being fastened to the horizontal beam, and a motion of rotation imparted, the portions of the hoop farthest from the axis will be most affected by centrifugal force, and the hoop will assume an elliptical form.

If a sponge, filled with water, be attached to one of the arms of a whirling machine, and motion of rotation imparted, the water will be thrown from the sponge. This principle has been used for drying clothes. An annular trough of copper is mounted on an axis by radial arms, and the axis connected with a train of wheelwork, by means of which it may be put in motion. The outer wall of the trough is pierced with holes for the escape of water, and a lid confines the articles to be dried. To use this instrument, the linen, after being washed, is placed in the annular space, and a rapid rotation imparted to the machine. The linen is thrown against the outer wall of the instrument, and the water, urged by the centrifugal force, escapes through the holes. Sometimes as many as 1,500 revolutions per minute are given to the drying machine, in which case, the drying process is very rapid and very perfect.

If a body revolve with sufficient velocity, it may happen that the centrifugal force generated will be greater than the force of cohesion that binds the particles together, and the body be torn asunder. It is a common occurrence for large grindstones, when put into rapid rotation, to burst, the fragments being thrown away from the axis, and often producing much destruction.

When a wagon, or carriage, is driven round a corner, or is forced to run on a circular track, the centrifugal force is often sufficient to throw loose articles from the vehicle, and

even to overthrow the vehicle itself. When a car on a railroad track is forced to turn a sharp curve, the centrifugal force throws the cars against the rail, producing a great amount of friction. To obviate this difficulty, it is customary to raise the outer rail, so that the resultant of the centrifugal force, and the force of gravity, shall be perpendicular to the plane of the rails.

Form of the Surface of a Revolving Liquid.

127. If a vessel of water be made to revolve about a vertical axis, the inner particles recede from the axis on account of the centrifugal force, and are heaped up about the sides of the vessel, imparting a concave form to the upper surface. The concavity becomes greater as the angular velocity is increased.

To determine the form of the concave surface, we assume the principle yet to be demonstrated, viz.: that the resultant action on any point of the free surface is normal to that surface. Let the figure represent a section made by a plane through the axis of rotation supposed vertical. Let BAC be the section of the upper surface of the water, A its lowest point, and P any other point. Take A as the origin of co-ordinates, the axis of Y being vertical, and the axis of X horizontal, and denote the co-ordinates of P by x and y.

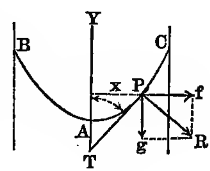

Fig. 112.

The material point P is urged horizontally by the centrifugal force Pf, whose acceleration (Eq. 196) is equal to $x\omega^2$, and it is urged downward by the force of gravity Pg, whose acceleration is g. The resultant of these forces, PR, is, from what precedes, normal to the curve APC. Because Pf is perpendicular to AY, and PR to the tangent PT, the angle fPR is equal to YTP; but the tangent of fPR is equal to

$\frac{g}{x\omega^2}$, and the tangent of YTP is (Calc., p. 12) equal to $\frac{dx}{dy}$. Equating these values and clearing of fractions, we have,

$$\omega^2 x\, dx = g\, dy. \quad . \quad . \quad . \quad . \quad (199)$$

Integrating (199), observing that the arbitrary constant under the given conditions is 0, we have,

$$\frac{\omega^2 x^2}{2} = gy, \quad \text{or} \quad x^2 = \frac{2g}{\omega^2}x, \quad . \quad . \quad . \quad . \quad . \quad (200)$$

which is the equation of a parabola whose axis coincides with the axis of revolution.

Hence, *the surface is a paraboloid of revolution whose axis is the axis of revolution.*

Centrifugal Force at Points of the Earth's Surface.

128. Let it be required to determine the centrifugal force at different points of the earth's surface, due to rotation on its axis.

Suppose the earth spherical. Let A be a point on the surface, PQP' a meridian section through A, PP' the axis, RQ the equator, and AB, perpendicular to PP', the radius of the parallel of latitude through A. Denote the radius of the earth by r, the radius of the parallel through A by r', and the latitude of A, or the angle QCA, by l. The time of revolution being the same for every point on the earth's surface, the velocities of Q and A will be to each other as their distances

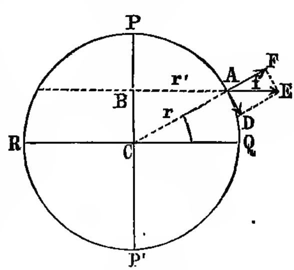

Fig. 113.

from the axis. Denoting these velocities by v and v', we have,

$$v : v' :: r : r',$$

whence,

$$v' = \frac{vr'}{r}.$$

But from the right-angled triangle, CAB, since the angle at A is equal to l, we have,

$$r' = r \cos l.$$

Substituting this value of r' in the value of v', and reducing, we have,

$$v' = v \cos l.$$

If we denote the centrifugal force at the equator by f, we have,

$$f = \frac{v^2}{r}. \quad \ldots\ldots \quad (201)$$

In like manner, if we denote the centrifugal force at A by f', we have,

$$f' = \frac{v'^2}{r'}.$$

Substituting for v' and r' their values, previously deduced, we get,

$$f' = \frac{v^2 \cos l}{r}. \quad \ldots\ldots \quad (202)$$

Combining equations (201) and (202), we find,

$$f : f' :: 1 : \cos l, \qquad \therefore \quad f' = f \cos l. \quad \ldots\ldots \quad (203)$$

That is, *the centrifugal force at any point on the earth's surface, is equal to the centrifugal force at the equator, multiplied by the cosine of the latitude.*

Let AE, perpendicular to PP', represent f', and resolve it into two components, one tangential, and the other normal to the meridian section. Prolong CA, and draw AD perpendicular to it at A. Complete the rectangle FD on AE as a diagonal. Then will AD be the tangential and AF the normal component. In the right-angled triangle, AFE, the angle at A is equal to l. Hence,

$$FE = AD = f' \sin l = f \cos l \sin l = \frac{f \sin 2l}{2} \quad \ldots \ldots (204)$$

$$AF = f' \cos l = f \cos^2 l. \quad \ldots \ldots (205)$$

From (204), we see that the tangential component is 0 at the equator, goes on increasing till $l = 45°$, where it is a maximum, and then goes on decreasing till the latitude is 90°, when it again becomes 0.

The effect of the tangential component is to heap up the particles of the earth about the equator, and, were the earth in a fluid state, this process would go on till the effect of the tangential component was counterbalanced by the component of gravity acting down the inclined plane thus formed, when the particles would be in equilibrium.

The higher analysis shows that the form of equilibrium is that of an oblate spheroid, differing but slightly from that which our globe is found to possess by actual measurement.

From equation (205), we see that the normal component of the centrifugal force varies as the square of the cosine of the latitude.

This component is directly opposed to gravity, and, consequently, tends to diminish the apparent weight of all bodies on the surface of the earth. The value of this component is greatest at the equator, and diminishes toward the poles, where it is 0. From the action of the normal component of the centrifugal force, and because the flattened form

of the earth due to the tangential component brings the polar regions nearer the centre of the earth, the measured force of gravity ought to increase in passing from the equator toward the poles. This is found to be the case.

The radius of the earth at the equator is about 3962.8 miles, which, multiplied by 2π, will give the entire circumference of the equator. If this be divided by the number of seconds in a day, 86,400, we find the value of v. Substituting this value of v and that of r just given, in equation (201), we find,

$$f = 0.1112\,ft.,$$

for the centrifugal force at the equator. If this be multiplied by the square of the cosine of the latitude of any place, we have the value of the normal component of the centrifugal force at that place.

If the earth were to revolve 17 times as rapidly as it now does, the centrifugal force at the equator would be equal to $0.1112\,ft. \times 289$, or to $32.1368\,ft.$, that is, the centrifugal force at the equator would be very nearly equal to g. In that case, the apparent weight of a body at the equator would be equal to 0.

Elevation of the Outer Rail of a Curved Track.

129. To find the elevation of the outer rail, so that the resultant of the weight and centrifugal force shall be perpendicular to the line joining the rails, assume a cross section through the centre of gravity, G. Take the horizontal, GA, to represent the centrifugal force, and GB to represent gravity. Construct their resultant, GC. Then must DE be perpendicular to GC.

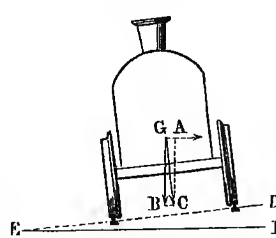

Fig. 114.

Denote the velocity of the car by v, the radius of the curved track by r, the force of gravity by g, and the angle, DEF, or its equal, BGC, by α. From the right-angled triangle, GBC, we have,

$$\tan \alpha = \frac{BC}{GB}.$$

But BC, which is equivalent to GA, is equal to $\frac{v^2}{r}$, and GB is equal to g; hence,

$$\tan \alpha = \frac{v^2}{gr}.$$

Denoting the distance between the rails, by d, and the elevation of the outer rail above the inner one, by h, we have,

$$\tan \alpha = \frac{h}{d}, \textit{ very nearly.}$$

Equating the two values of $\tan \alpha$, we have,

$$\frac{h}{d} = \frac{v^2}{gr}, \quad \therefore h = \frac{dv^2}{gr} \quad . \quad . \quad . \quad . \quad . \quad (206)$$

Hence, the elevation of the outer rail varies as the square of the velocity directly, and as the radius of the curve inversely.

It is obvious that the elevation ought to be different for different velocities, which, from the nature of the case, is impossible. The correction is, therefore, made for some assumed velocity, and then such a form is given to the tire of the wheels as to partially correct for other velocities.

The Conical Pendulum.

130. **A conical pendulum** consists of a solid ball attached to one end of a rod, the other end of which is connected, by means of a hinge-joint, with a vertical axle.

When the axle is put in motion, the centrifugal force generated in the ball causes it to recede from the axis, until an equilibrium is established between the weight of the ball, the centrifugal force, and the tension of the connecting rod. When the velocity is constant, the centrifugal force will be constant, and the centre of the ball will describe a horizontal circle, whose radius will depend upon the velocity. Let it be required to determine the time of revolution.

Let BD be the vertical axis, A the ball, B the hinge-joint, and AB the connecting rod, whose mass is so small, that it may be neglected, in comparison with that of the ball.

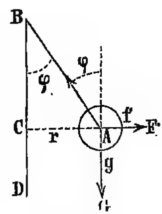

Fig. 115.

Denote the required time of revolution, by t, the length of the arm, by l, the acceleration due to the centrifugal force, by f, and the angle ABC, by ϕ. Draw AC perpendicular to BD, and denote AC, by r, and BC, by h.

From the triangle, ABC, we have, $r = h \tan \phi$; and since r is the radius of the circle described by A, the distance passed over by A, in the time t, is equal to $2\pi r$, or, $2\pi h \tan \phi$. Denoting the velocity of A, by v, we have,

$$v = \frac{2\pi h \tan \phi}{t}.$$

But the centrifugal force is equal to the square of the velocity, divided by the radius; hence,

$$f = \frac{4\pi^2 h \tan \phi}{t^2} \quad . \; . \; . \; . \; . \quad (207)$$

The forces that act on A, are the centrifugal force, in the direction AF, the force of gravity, in the direction AG, and the resistance of the connecting rod, in the direction AB. In order that the ball may remain at an invariable distance

from the axis, these must be in equilibrium. Hence, Eq. (16),

$$g : f :: \sin BAF :: \sin BAG;$$

but, $\sin BAF = \sin (90° + \phi) = \cos \phi$;

and, $\sin BAG = \sin (180° - \phi) = \sin \phi$.

We have, therefore,

$$g : f :: \cos \phi : \sin \phi, \qquad \therefore f = g \tan \phi.$$

Equating these values of f, we have,

$$\frac{4\pi^2 h \tan \phi}{t^2} = g \tan \phi.$$

Solving with respect to t,

$$t = 2\pi \sqrt{\frac{h}{g}} \quad . \ . \ . \ . \ . \quad (208)$$

That is, *the time of revolution, is equal to the time of a double vibration of a pendulum whose length is h.*

The Governor.

131. The principle of the conical pendulum is employed in the **governor**, a machine attached to engines, to regulate the supply of motive force.

AB is a vertical axis connected with the machine near its working-point, and and revolving with a velocity proportional to that of the working-point; FE and GD are arms turning about AB, and bearing heavy balls, D and E, at their extremities; these bars are united by hinge-joints with two other bars at G and F, and also to a ring at H, that is free to slide up and down the shaft.

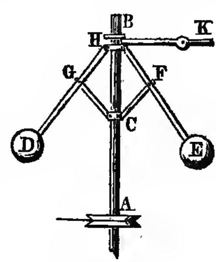

Fig. 116.

The ring, H, is connected with a lever, HK, that acts on the throttle valve in the pipe that admits steam to the cylinder.

When the shaft revolves, the centrifugal force causes the balls to recede from the axis, and the ring, H, is depressed; and when the velocity has become sufficiently great, the lever closes the valve. If the velocity slackens, the balls approach the axis, and the ring, H, ascends, opening the valve. In any given case, if we know the velocity required at the working-point, we can compute the required angular velocity of the shaft, and, consequently, the value of t. This value of t, substituted in equation (208), gives the value of h. We may, therefore, properly adjust the ring, and the lever, HK.

EXAMPLES.

1. A ball weighing 10 *lbs.* is whirled round in a circle whose radius is 10 feet, with a velocity of 30 feet. What is the acceleration due to centrifugal force? *Ans.* 90 *ft.*

2. In the preceding example, what is the tension on the cord that restrains the ball? *Ans.* $t = 28$ *lbs., nearly.*

3. A body is whirled round in a circular path whose radius is 5 feet, and the centrifugal force is equal to the weight of the body. What is the velocity of the moving body? *Ans.* $v = 12.7$ *ft.*

4. In how many seconds must the earth revolve that the centrifugal force at the equator may counterbalance the force of gravity, the radius of the equator being 3962.8 miles? *Ans.* $t = 5{,}068$ *secs.*

5. A body is placed on a horizontal plane, which is made to revolve about a vertical axis, with an angular velocity of 2 feet. How far must the body be situated from the axis that it may be on the point of sliding outward, the coefficient of friction between the body and plane being equal to .6? *Ans.* $r = 4.825$ *ft.*

6. What must be the elevation of the outer rail of a track, the radius being 3960 *ft.*, the distance between the rails 5 feet, and the velocity of the car 30 miles per hour, that there may be no lateral thrust?

Ans. 0.076 *ft.*, or 0.9 *in., nearly.*

7. The distance between the rails is 5 feet, the radius of the curve 600 feet, and the height of the centre of gravity of the car 5 feet. What

velocity must the car have that it may be on the point of being overturned by the centrifugal force, the rails being on the same level?

Ans. $v = 98$ *ft.*, or $66\frac{3}{4}$ *miles per hour.*

8. A body revolves uniformly in a circle whose radius is 5 *ft.*, and with such a velocity as to complete a revolution in 5 seconds. What is the acceleration due to the centripetal force? *Ans.* $\frac{4}{5}\pi^2$.

9. A body weighing 1 *lb.*, is whirled around horizontally, being retained by a string, whose length is 6 feet. What is the time of revolution when the tension of the string is 3 *lbs.*? *Ans.* $2\pi\sqrt{\frac{2}{g}}$ seconds.

Moment of Inertia.

132. The **moment of inertia** of a body with respect to an axis, is the *algebraic* sum of the products obtained by multiplying the mass of each element of the body by the square of its distance from the axis. Denoting the moment of inertia with respect to any axis by K, the mass of any element of the body by m, and its distance from the axis by r, we have, from the definition,

$$K = \Sigma(mr^2). \quad . \quad . \quad . \quad . \quad . \quad (209)$$

If we denote the mass of any element by m and its distance from the axis by r, the velocity of m when the angular velocity of the body is 1 will be equal to r and the momentum of m will be mr: but the particle is moving in a direction that is perpendicular to r; hence the moment of the momentum with respect to the axis is mr^2. The algebraic sum of the moments of the momenta of all the elements of the body will be equal to $\Sigma(mr^2)$, which is the expression for the moment of inertia of the body, that is, *the moment of inertia of a body with respect to any axis is the algebraic sum of moments of the momenta of all its elements, with respect to the same axis, when the body is revolving with an angular velocity equal to* 1.

If we denote the algebraic sum of the moments of the momenta of all the elements of a body by L, the angular velocity being ω, it is obvious that we shall have,

$$L = \Sigma (mr^2) \omega = K\omega. \quad . \quad . \quad . \quad . \quad . \quad (210)$$

Moment of Inertia with respect to Parallel Axes.

133. The moment of inertia of a body varies with the position of the axis with respect to which it is taken.

To investigate the law of variation, let AB represent any section of the body by a plane perpendicular to the axis; C, the point in which this plane cuts the axis; and G, the point in which it cuts a parallel axis through the centre of gravity. Let P be any element of the body, whose mass is m, and denote PC by r, PG by s, and CG by k.

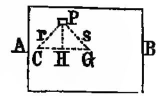

Fig. 117.

From the triangle CPG, according to a principle of Trigonometry, we have,

$$r^2 = s^2 + k^2 - 2sk \cos CGP.$$

Substituting in (209) and separating the terms, we have,

$$K = \Sigma (ms^2) + \Sigma (mk^2) - 2\Sigma (msk \cos CGP).$$

Or, since k is constant, and $\Sigma (m) = M$, the mass of the body, we have,

$$K = \Sigma (ms^2) + Mk^2 - 2k\Sigma (ms \cos CGP).$$

But $s \cos CGP = GH$, the lever arm of the mass m, with respect to the axis through the centre of gravity. Hence, $\Sigma (ms \cos CGP)$ is the algebraic sum of the moments of all the particles of the body with respect to the axis through the

centre of gravity; but, from the principle of moments, this is equal to 0. Hence,

$$K = \Sigma(ms^2) + Mk^2. \quad . \quad . \quad . \quad . \quad . \quad (211).$$

The first term of the second member is the moment of inertia, with respect to the axis through the centre of gravity.

Hence, *the moment of inertia of a body with respect to any axis, is equal to the moment of inertia with respect to a parallel axis through the centre of gravity,* plus *the mass of the body into the square of the distance between the two axes.*

The moment of inertia for any system of parallel axes is least possible when the axis passes through the centre of gravity. If any number of parallel axes be taken at equal distances from the centre of gravity, the moment of inertia with respect to each will be the same.

Equation (211) enables us to find the moment of inertia with respect to any axis when we know the moment of inertia with respect to any parallel axis. If the first axis does not pass through the centre of gravity of the body, we pass to a parallel axis by diminishing the given moment of inertia by Mk^2; we then pass to the second axis by increasing the last result by Mk'^2; k and k' are the distances from the first and second axes respectively, to the parallel axis through the centre of gravity of the body.

Polar Moment of a Plane Surface.

134. The **polar moment of inertia** is the moment of inertia of a plane surface with respect to an axis perpendicular to the plane.

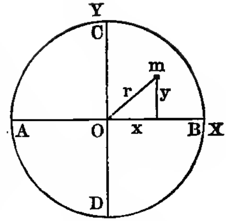

Fig. 118.

Let $ACBD$ be any plane area, and O the point in which it cuts an axis perpendicular to it: through O draw any two rectangular axes, OX and OY. Let m be an infinitesimal element whose co-ordinates are x and y, and whose distance from O is r. Then will the polar moment of inertia of the area be given by the equation,

$$K = \Sigma (mr^2). \quad . \quad . \quad . \quad . \quad . \quad (212)$$

But, $r^2 = y^2 + x^2$: substituting in (212) and separating the terms, we have,

$$K = \Sigma (my^2) + \Sigma (mx^2). \quad . \quad . \quad . \quad . \quad . \quad (213)$$

The first term of the second member of (213) is the moment of inertia of the given area with respect to the axis of X, and the second term is its moment of inertia with respect to the axis of Y; denoting the former by I_x and the latter by I_y, we have,

$$K = I_x + I_y. \quad . \quad . \quad . \quad . \quad . \quad (214)$$

Hence, *the polar moment of inertia of a plane area is equal to the sum of the moments of inertia of the area with respect to any two rectangular axes in the plane of the area passing through the foot of the polar axis.*

Experimental Determination of Moment of Inertia.

135. When a body can be handled conveniently, its moment of inertia may be found *experimentally* as follows: Make the axis horizontal, and allow the body to vibrate about it, as a compound pendulum. Find the time of a single vibration, and denote it by t. This value of t, in equation

(184), makes known the value of l. Determine the centre of gravity, and denote its distance from the axis by k. Find the mass of the body, and denote it by M.

We have, from equation (190),

$$Mkl = \Sigma\,(mr^2) = K. \quad . \quad . \quad . \quad . \quad . \quad (215)$$

Substitute for M, l, and k, the values already found, and the value of K will be the moment of inertia, with respect to the assumed axis. Subtract from this the value of Mk^2, and the remainder will be the moment of inertia with respect to a parallel axis through the centre of gravity.

When a body is homogeneous and of regular figure, its moment of inertia is most readily found by means of the calculus.

To make formula (209) suitable to the application of the calculus, we have simply to change the sign of summation, Σ, to that of integration, $\int$, and to replace m by dM and r by x. This gives,

$$K = \int x^2 dM \quad . \quad . \quad . \quad . \quad . \quad . \quad (216)$$

Moment of Inertia of a Straight Line.

136. Let AB represent a physical straight line, G its centre of gravity, and E any element limited by planes at right angles to its length and infinitely near to each other. Denote the mass of the line by M, its length by $2l$, the distance GE by x, and the length of the element by dx; also denote the mass of the element by dM: we shall then have,

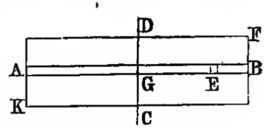

Fig. 119.

$$M : dM :: 2l : dx, \quad \text{or,} \quad dM = \frac{Mdx}{2l};$$

Substituting this in (216), and taking the integral between the limits $-l$ and $+l$, we have,

$$K = \int_{-l}^{+l} \frac{M}{2l} x^2 dx = M\frac{l^2}{3} \quad . \quad . \quad . \quad . \quad . \quad (217)$$

That is, *the moment of inertia of a right line with respect to a perpendicular axis through its centre of gravity is equal to the mass of the line multiplied by one third of the square of half its length.*

For a parallel axis whose distance from G is d, the moment of inertia being denoted by K', we have,

$$K' = M\left(\frac{l^2}{3} + d^2\right) \quad . \quad . \quad . \quad . \quad . \quad (218)$$

For a parallel axis through the end of the line we have $d = l$, and (218) becomes,

$$K' = M\left(\frac{l^2}{3} + l^2\right) = \frac{4}{3}Ml^2 \quad . \quad . \quad . \quad . \quad . \quad (219)$$

Formulas (217), (218), and (219) are entirely independent of the breadth of the filament AB: they will therefore hold good when the filament is replaced by the rectangle KF. In this case M represents the mass of the rectangle, $2l$ is its length, and d is the distance of the centre of gravity of the rectangle from an axis parallel to one of its ends.

Moment of Inertia of a Thin Circular Plate.

137. In the *first* place, let us find the polar moment of inertia, the axis being perpendicular to the plane of the circle at its centre. From the centre with a radius x describe a circle, and again with the radius $x + dx$ describe a concentric circle; all points of the included ring are equally distant

from the axis, and consequently its moment of inertia is equal to its mass *multiplied* by the square of x. Denoting the mass of the circle by M and the mass of the elementary ring by dM, we have,

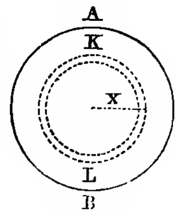

Fig. 120.

$$M : dM :: \pi r^2 : 2\pi x\, dx,$$

from which we deduce,

$$dM = \frac{M}{\pi r^2} \times 2\, \pi x\, dx = \frac{2M}{r^2} x\, dx.$$

Substituting in (216), and integrating between the limits $x = 0$ and $x = r$, we have,

$$K = \frac{2M}{r^2} \int_0^r x^3 dx = \frac{Mr^2}{2} \quad . \; . \; . \; . \; . \quad (220)$$

For a parallel axis at a distance d from the axis through the centre, we have,

$$K' = M\left(\frac{r^2}{2} + d^2\right) \quad . \; . \; . \; . \; . \quad (221)$$

In the *second* place, to find the moment of inertia of a circle with respect to any diameter, we observe that the moment of inertia is the same for all diameters; hence, from (214), we have,

$$K = \frac{Mr^2}{4} \quad . \; . \; . \; . \; . \quad (222)$$

And for a parallel axis at the distance d,

$$K' = M\left(\frac{r^2}{4} + d^2\right) \quad . \; . \; . \; . \; . \quad (223)$$

To find the moment of inertia of a circular ring with reference to an axis perpendicular to the plane of the ring through its centre, we integrate (220) between the limits r' and r, r' being the inner radius of the ring and r the outer one. In this case, we have,

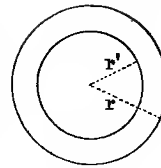

Fig. 121.

$$K = \frac{2M}{r^2}\left(\frac{r^4}{4} - \frac{r'^4}{4}\right) = \frac{M}{2r^2}(r^2 - r'^2)(r^2 + r'^2) \quad . \quad . \quad . \quad . \quad (a)$$

in which M is the mass of the outer circle. If we denote the mass of the ring by M', we have,

$$M : M' :: r^2 : r^2 - r'^2, \quad \text{or} \quad \frac{M}{r^2} = \frac{M'}{r^2 - r'^2}.$$

Substituting in (a), and reducing, we have,

$$K = M'\left(\frac{r^2 + r'^2}{2}\right) \quad . \quad . \quad . \quad . \quad . \quad (224)$$

For a parallel axis, we have,

$$K = M'\left(\frac{r^2 + r'^2}{2} + d^2\right) \quad . \quad . \quad . \quad . \quad . \quad (225)$$

It is to be noted that the preceding expressions are entirely independent of the thickness of the circular plate; hence, they are applicable to cylinders of any length (either solid or hollow), when their axes are parallel to the axis of rotation.

Let the student solve the same problem, using polar co-ordinates.

Moment of Inertia of any Solid of Revolution.

138. Let PAQ be the meridian section of a solid of revolution, and take the axis OX of the solid as the axis of X. Let

OZ, perpendicular to OX, be the axis, with respect to which the moment of inertia is to be found.

At a distance x from the origin pass a plane PQ perpendicular to OX, and at a distance $x + dx$ pass a parallel plane. These planes will include an elementary slice of the solid whose volume is $\pi y^2 dx$, y being the ordinate of the meridian curve corresponding to the abscissa x. If we denote the entire volume by V, its mass by M, and the mass of the slice by dM, we have,

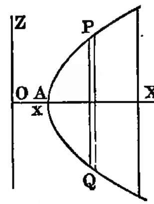

Fig. 122.

$$M : dM :: V : \pi y^2 dx, \quad \text{or} \quad dM = \frac{\pi M}{V} y^2 dx.$$

The moment of inertia of the slice with respect to OZ, may be found from Equation (223), by making $M = dM$, $r = y$, and $d = x$: calling this moment of inertia dK, we have,

$$dK = \frac{\pi M}{V}\left(\frac{y^4}{4} + x^2y^2\right)dx.$$

Whence, by integration,

$$K = \frac{\pi M}{V}\int\left(\frac{y^4}{4} + x^2y^2\right)dx, \quad \ldots\ldots \quad (226)$$

the limits being taken to include the entire body.

Moment of Inertia of a Cylinder.

139. Let the axis of the cylinder coincide with the axis of X, and let the axis OZ be taken through the centre of gravity of the cylinder; denote the length of the cylinder by $2l$, and the radius of its cross section by r.

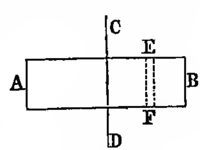

Fig. 123.

Then will $y = r$, and $V = 2\pi r^2 l$: substituting in (226), and integrating between the limits $-l$ and $+l$, we have,

$$K = \frac{M}{2r^2l}\int_{-l}^{+l}\left(\frac{r^4}{4} + r^2x^2\right)dx = M\left(\frac{r^2}{4} + \frac{l^2}{3}\right) \quad . \quad . \quad . \quad . \quad . \quad (227)$$

For an axis parallel to CD, we have,

$$K' = M\left(\frac{r^2}{4} + \frac{l^2}{3} + d^2\right) \quad . \quad . \quad . \quad . \quad . \quad (228)$$

Moment of Inertia of a Cone.

140. Let the axis of the cone coincide with the axis of X, and let OZ be perpendicular to it at the vertex of the cone; denote the height of the cone by h, the radius of its base by r, and its mass by M.

We shall have $y = \frac{r}{h}x$, and $V = \frac{1}{3}\pi r^2 h$: substituting, in (226), and taking the integral between the limits 0 and h, we have,

$$K = \frac{3M}{r^2h}\int_0^h\left(\frac{r^4}{4h^4}x^4 + \frac{r^2}{h^2}x^4\right)dx = \frac{3M}{20}\left(r^2 + 4h^2\right) \quad . \quad . \quad . \quad . \quad (229)$$

To find the moment of inertia when OZ passes through the centre of gravity of the cone, we subtract from (229) the quantity $M \times \left(\frac{3}{4}h\right)^2$, whence,

$$K = M\left(\frac{3}{20}r^2 + \frac{3}{5}h^2 - \frac{9}{16}h^2\right) = \frac{3}{20}M\left(r^2 + \frac{1}{4}h^2\right) . \quad . \quad . \quad . \quad . \quad (230)$$

Moment of Inertia of a Sphere.

141. Let the axis of X be taken to coincide with a diameter of the sphere, and let the axis of Z be perpendicular to it at its left hand extremity; denote the radius of the sphere by r, and its mass by M.

We shall then have, $y^2 = 2rx - x^2$, and $V = \frac{4}{3}\pi r^3$; substi-

tuting, in (226), and taking the integral between the limits 0 and $2r$, we have,

$$K = \frac{3M}{4r^3}\int_0^{2r}\left(\frac{4r^2x^2 - 4rx^3 + x^4}{4} + 2rx^3 - x^4\right)dx,$$

or,

$$K = \frac{3M}{4r^3}\int_0^{2r}\left(r^2x^2 + rx^3 - \frac{3}{4}x^4\right)dx = \frac{7}{5}Mr^2 \quad . \quad . \quad . \quad . \quad . \quad (231)$$

For a parallel axis through the centre we subtract, from (231), the quantity, Mr^2, which gives,

$$K' = \frac{2}{5}Mr^2 \quad . \quad . \quad . \quad . \quad . \quad (232)$$

The origin might have been taken at the centre in the first place, in which case we should at once have deduced equation (232), from which we readily deduce equation (231).

Moment of Inertia of a Solid with Respect to Its Axis.

142. Recurring to Article 138, let us suppose the circular slice, PQ, to revolve around the axis OX: its moment of inertia, which is the *differential* of the moment of inertia of the solid with respect to OX, may be found, from (220), by making $r = y$, and M equal to the mass of the slice, Art. 138; making these substitutions, and calling the result dK, we have,

$$dK = \frac{\pi M}{V} \cdot \frac{y^4}{2}dx.$$

Whence, by integration,

$$K = \frac{\pi M}{2V}\int y^4dx \quad . \quad . \quad . \quad . \quad . \quad (233)$$

In which y is the ordinate of the meridian curve.

Moment of Inertia of a Cylinder.

143. To find the moment of inertia of a cylinder with respect to its axis, we assume the co-ordinate axes, as in Art. 139.

We then have, $V = \pi r^2 \times 2l$ and $y = r$. Substituting, in (233), and integrating from $-l$ to $+l$, we have,

$$K = \frac{M}{4lr^2} \int_{-l}^{+l} r^4 dx = \frac{M}{2} r^2, \quad . \quad . \quad . \quad . \quad . \quad (234)$$

a result that corresponds with (220).

Moment of Inertia of a Cone.

144. Assume the position of the axis, and the notation of Art. 140. We then have $V = \frac{1}{3}\pi r^2 h$, and $y = \frac{r}{h} x$: substituting, in (233), and taking the integral between the limits 0 and h, we have,

$$K = \frac{3}{2} \cdot \frac{M}{r^2 h} \int_0^h \frac{r^4}{h^4} x^4 dx = \frac{3}{10} \mathrm{M} r^2 \quad . \quad . \quad . \quad . \quad . \quad (235)$$

Moment of Inertia of a Sphere.

145. Let the axis OX be a diameter of the sphere, and let OZ be perpendicular to it at its left hand extremity. If we assume the notation already employed, we have, $V = \frac{4}{3}\pi r^3$, and $y^2 = 2rx - x^2$; substituting, in (233), taking the integral between the limits 0 and $2r$, we have,

$$K = \frac{3}{8} \frac{M}{r^3} \int_0^{2r} (4r^2x^2 - 4rx^3 + x^4)\, dx = \frac{2}{5} M r^2, \quad . \quad . \quad . \quad . \quad . \quad (236)$$

which corresponds to equation (232).

Moment of Inertia with Respect to Axis of Symmetry.

146. Let $APBQ$ be an ellipse, AB its transverse, and CD its conjugate axis. Let the origin of co-ordinates be taken at the centre O; at a distance from O equal to x, draw a double ordinate, PQ, and at a distance, $x + dx$, draw a parallel double ordinate: the area of the included filament is equal to $2ydx$; denoting the mass of this filament by dM, that of the ellipse being denoted by M, and calling the semi-axes of the curve a and b, we have,

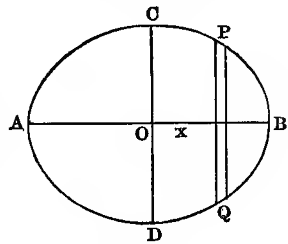

Fig. 124.

$$M : dM :: \pi ab : 2ydx, \quad \text{or} \quad dM = \frac{2Mydx}{\pi ab} \quad . \quad . \quad . \quad . \quad . \quad (237)$$

The moment of inertia of PQ with respect to the axis AB, which is the differential of the entire moment of inertia, is given by Equation (217), by making M equal to the value of dM, Eq. 237, and $l = y$. Substituting these values, and integrating, we have,

$$K = \frac{2M}{3\pi ab}\int y^3 dx \quad . \quad . \quad . \quad . \quad . \quad (238)$$

From the equation of the ellipse, referred to its centre, we have,

$$y = \frac{b}{a}(a^2 - x^2)^{\frac{1}{2}}.$$

Substituting in (238), reducing, and taking the integral between the limits $-a$ and $+a$, we have,

$$K = \frac{2}{3}\frac{Mb^2}{\pi a^4}\int_{-a}^{+a}(a^2 - x^2)^{\frac{3}{2}}dx \quad . \quad . \quad . \quad . \quad . \quad (239)$$

Applying formula B twice, and completing the integration by the *sine formula*, Calc., p. 107, we finally obtain

$$K = \tfrac{1}{4}Mb^2 \quad . \quad . \quad . \quad . \quad . \quad (240)$$

In like manner, with respect to the conjugate axis, we find the moment of inertia to be

$$K = \tfrac{1}{4}Ma^2 \quad . \quad . \quad . \quad . \quad . \quad (241)$$

By a similar course of investigation we can find the moment of inertia of any plane area with respect to an axis of symmetry.

Centre and Radius of Gyration.

147. The **centre of gyration** of a body with reference to any axis is a point lying on a perpendicular to this axis through the centre of gravity, such that if the entire mass of the body were concentrated at it, its moment of inertia would not be changed. The distance from the centre of gyration to the axis is called the **radius of gyration.**

Let M denote the mass of the body, and h its radius of gyration; then will the moment of inertia of the concentrated mass, with respect to the axis, be equal to Mh^2; but this must, by definition, be equal to the moment of inertia of the body with respect to the same axis, or to $\Sigma\,(mr^2)$; hence,

$$Mh^2 = \Sigma\,(mr^2), \quad \text{or} \quad h = \sqrt{\frac{\Sigma\,(mr^2)}{M}} \quad . \quad . \quad . \quad . \quad . \quad (242)$$

Hence, *to find the radius of gyration with respect to an axis, we divide its moment of inertia with respect to that axis, by the mass, and then extract the square root of the result.*

Since M is constant for the same body, it follows that the radius of gyration will be the least possible when the moment

of inertia is the least possible, that is, when the axis passes through the centre of gravity. This minimum radius is called the **principal radius of gyration.** If we denote the principal radius of gyration by h_0, we shall have, for the straight line,

$$h_0 = \frac{l}{\sqrt{3}}, \quad \text{and} \quad h = \sqrt{\frac{l^2}{3} + d^2}, \quad . \quad . \quad . \quad . \quad . \quad (243)$$

and in like manner we may treat the other magnitudes that have been considered.

The straight line through the centre of gyration parallel to the axis with reference to which the centre of inertia is taken, is sometimes called the **axis of gyration.** If this line be revolved around the axis of rotation of the body, it will generate a cylinder ; if the mass of the body be concentrated in any manner on the surface of this cylinder, its moment of inertia will not be changed.

In a compound pendulum, the centre of gravity, the centre of oscillation, and the centre of gyration, all lie on the same straight line ; the last of these points lies between the other two. Equation (190) may be written

$$l = \frac{Mh^2}{Mk} = \frac{h^2}{k}, \quad \text{or} \quad kl = h^2, \quad \text{or} \quad h = \sqrt{kl} \quad . \quad . \quad . \quad . \quad . \quad (244)$$

Hence, h is a mean proportional between k and l. When any two of these quantities are given, the other one may be found by geometrical construction.

VII.—WORK AND ENERGY.—IMPACT.

Relation Between Work and Energy.

148. The terms **work** and **energy** are defined in Article 9. From that article and from Article 32 we see that the elementary quantity of work of a force is measured by the intensity of the force *multiplied* by the projection of the elementary path of its point of application on the direction of the force.

Let a force F be exerted on a free body whose mass is m, and suppose that the point of application passes over a distance k in the element of time dt. Let the distance k be projected on F, and denote the projection by ds. If we denote the elementary quantity of work by dQ, we shall have,

$$dQ = Fds.$$

But in this case we have, from Art. 92,

$$F = m\frac{d^2s}{dt^2}.$$

Substituting this value of F in the preceding equation, we have,

$$dQ = m\frac{d^2s}{dt^2}\,ds \quad . \; . \; . \; . \; . \quad (244)$$

Integrating Equation (244), we have,

$$Q = \tfrac{1}{2}m\frac{ds^2}{dt^2} + C = \tfrac{1}{2}mv^2 + C \quad . \; . \; . \; . \; . \quad (245)$$

If we suppose the body to start from rest, Q will be 0 when v is 0, and, consequently, $C = 0$; hence, we have,

$$Q = \tfrac{1}{2}mv^2 \quad . \quad . \quad . \quad . \quad . \quad (246)$$

That is, quantity of work performed by the force F, in generating the velocity v, is equal to *one half the mass of the body* multiplied by *the square of the velocity generated.*

The quantity $\tfrac{1}{2}mv^2$ represents a quantity of work that has been **stored up** in the body by virtue of the action of the force F, all of which will be **given out** whilst the body is coming to rest. Hence, $\tfrac{1}{2}mv^2$ represents the quantity of work that the moving body is capable of performing whilst coming to rest; in other words, it is the measure of the body's **kinetic energy**. The expression $\tfrac{1}{2}mv^2$ is also called the **living force** of the body.

To illustrate the relation between *work* and energy, let us consider the motion of a pendulum whose mass is concentrated at the centre of oscillation, hurtful resistances being neglected. In falling from the highest to the lowest point of its arc of vibration the force of gravity acts continually to increase the velocity, and the work that it performs is stored up in the pendulum: when it reaches the lowest point of its path the amount of work stored up will be equal to $\tfrac{1}{2}mv^2$, in which m is the mass of the pendulum and v the velocity due to the height through which it has fallen. By virtue of its kinetic energy the pendulum is carried past its lowest point and made to move along the ascending branch of its path; during this period of its motion it is working against the force of gravity, its velocity is continually decreasing, and finally, when it reaches a height equal to that from which it fell, the velocity becomes 0. The work *stored up* in descending is *given out* in ascending.

Discussion.

149. If we take the integral (245) between the limits v' and v'' and denote the corresponding quantity of work by Q', we have,

$$Q' = \tfrac{1}{2}m(v''^2 - v'^2) \quad . \quad . \quad . \quad . \quad . \quad (247)$$

In this expression v' is called the **initial velocity** and v'' the **terminal velocity.**

Let us suppose that the velocity of the body increases continuously from v' to v''; in this case Q' is *positive,* and work has been *stored up:* if we suppose that the velocity diminishes continuously from v' to v'', Q' will be *negative;* in this case work has been *given out.*

It may happen that the velocity in passing from v' to v'' is fluctuating, sometimes increasing, sometimes diminishing. When the velocity is increasing work is being *stored up,* when the velocity is diminishing work is being *given out.* In this case the value of Q', in (247), represents the *aggregate* quantity of work that is performed during the interval corresponding to the change of velocity from v' to v'', the work of external forces on the body being *positive,* and the work of the body in overcoming resistances being *negative.* If the terminal velocity v'' is equal to v', the work *stored up* will be equal to the work *given out.*

Bodies Moving Vertically Under the Influence of Gravity.

150. Let a body whose weight is w fall from rest through a height h; the work of gravity during the fall will be equal to wh; but, from Eq. (108), we have, $h = \dfrac{v^2}{2g}$; denoting the work by Q, we have,

$$Q = wh = \tfrac{1}{2}\frac{w}{g}v^2 = \tfrac{1}{2}mv^2, \quad . \quad . \quad . \quad . \quad . \quad (248)$$

a result which conforms to Eq. (246).

In this case we see that a body whose weight is w, and which is free to fall through a height h, is capable of performing an amount of work during the fall which is equal to $\frac{1}{2}mv^2$. This capacity to perform work is called **potential energy**, and as we have just seen it is equal to *one half the mass of the body* multiplied by *the square of the velocity due to the height through which the body is free to fall.*

When the body starts from rest it has no kinetic energy, but its potential energy is equal to $\frac{1}{2}mv^2$; after the body has fallen through a space x, its kinetic energy will, from what precedes, become equal to $\frac{1}{2}m(2gx)$, $2gx$ being the square of the velocity due to the height x; at the same instant its potential energy will have become $\frac{1}{2}m \times 2g(h-x)$, the factor $2g(h-x)$ being the square of the velocity due to the height through which the body is now free to fall. Adding these results, we have,

$$\tfrac{1}{2}m \times 2gx + \tfrac{1}{2}m \times 2g(h-x) = \tfrac{1}{2}m \times 2gh = \tfrac{1}{2}mv^2 \quad . \quad . \quad (249)$$

Hence, we see that the sum of the potential energy and the kinetic energy is *constant*. During the fall all of the potential energy is converted into kinetic energy.

If we suppose the body to be projected vertically upward with a velocity v, it will start with a kinetic energy equal to $\frac{1}{2}mv^2$; as it rises, kinetic energy is being continually converted into potential energy, the sum of the two being constant; when it attains a height, h, due to the velocity of projection, all of its kinetic energy will have been converted into potential energy, which will then be equal to $\frac{1}{2}mv^2$.

Motion on a Curve whose Plane is Vertical.

151. Let a body whose mass is m fall down a curve in a vertical plane under the influence of gravity, hurtful resistances being neglected. If the body start from a state of rest,

the velocity generated in reaching any point will (Art. 104) be that due to the vertical height through which the body has fallen, and the kinetic energy will be equal to $\frac{1}{2}mv^2$. If the body is projected upward from this point with a velocity v, tangential to the path at the point, it will retrace its path, the circumstances of motion being exactly reversed.

In falling down the curve the potential energy of the body is being continually converted into kinetic energy, and in ascending the curve kinetic energy is being continually converted into potential energy. From what precedes we see that the work required to draw a body up an inclined plane, or up a curve, hurtful resistance being neglected, is the same as would be required to lift the body vertically through the height of the plane or of the curve.

If a body in a stable position, as a pyramid resting on its base, be overturned by any extraneous force, the quantity of work will be equal to the weight of the body, multiplied by the vertical height to which the centre of gravity must be raised before reaching its highest point. This product might be taken as the measure of the stability of a body.

Work on an Inclined Plane, Friction being Considered.

152. Let a body be drawn up an inclined plane whose height is h, and whose length is l, the weight of the body being denoted by w, the coefficient of friction by f, and the inclination of the plane by α.

The component of w, which is normal to the plane, is $w \cos \alpha$; hence, the entire friction is $fw \cos \alpha$, and the work of friction through the length of the plane is $fw \cos \alpha \times l$; but $l \cos \alpha$ is equal to b, the length of the base of the plane, and consequently the entire work of friction is equal to $fw \times b$, that is, it is equal to the work of friction in drawing the body through the length of the base. Adding to this the

work of drawing the body up the plane when friction is neglected, which is wh, we have the following principle:

The work expended in drawing a body up a plane, when friction is taken into account, is equal to the work required to draw the body horizontally through the length of the base plus *the work required to lift the body through the height of the plane.*

This corresponds to the result deduced in Article 73.

Work of Raising a System of Bodies Vertically.

153. To find the work required to raise several bodies through different heights.

Let us consider the case when there are three bodies whose weights are w, w', and w''. Denote the distances of these bodies from a fixed horizontal plane by a, b, and c respectively. Then from Equation (30) the distance of the centre of gravity of the system from the plane of reference will be given by the equation,

$$z' = \frac{wa + w'b + w''c}{w + w' + w''} \quad . \quad . \quad . \quad . \quad . \quad (250)$$

Now, suppose the three bodies to be raised vertically through the distances a', b', and c' respectively. The distance of the centre of gravity of the system in its new position from the plane of reference will be given by the equation,

$$z'' = \frac{w\,(a + a') + w'\,(b + b') + w''\,(c + c')}{w + w' + w''} \quad . \quad . \quad . \quad . \quad . \quad (251)$$

Subtracting (250) from (251), member from member, we have,

$$z'' - z' = \frac{wa' + w'b' + w''c'}{w + w' + w''}, \quad . \quad . \quad . \quad . \quad . \quad (252)$$

or, clearing of fractions,

$$(w + w' + w'')(z'' - z') = wa' + w'b' + w''c' \quad . \quad . \quad . \quad . \quad . \quad (253)$$

The *second* member represents the aggregate work required to raise the separate weights to the required heights; the first member represents the work required to raise a weight equal to the aggregate weight of the bodies through the distance passed over by their centre of gravity.

Because, we may reason in like manner, when there are any number of bodies, we have the following principle:

The work required to raise several weights through different heights, is the same as that required to raise a weight equal to their sum through a distance equal to that passed over by their centre of gravity.

NOTE.—Because energy is always equivalent to a quantity of work, the *unit of energy* is a *pound foot.*

EXAMPLES.

1. What amount of work is required to raise 500 *lbs.* to the height of 5 yards? *Ans.* 7500 units, or 7500 *lb. ft.*

2. To what height can 2240 *lbs.* be raised by the expenditure of 5,600 units of work? *Ans.* 2.5 *ft.*

3. What weight can be raised to the height of 25 feet by 224,000 units of work? *Ans.* 8,960 *lbs.*

4. What is the effective horse-power of an engine which raises 80 cubic feet of water per minute from the depth of 360 feet, a cubic foot of water weighing 62 *lbs.* *Ans.* 54.11 horse-power.

5. What must be the effective horse-power to raise the same quantity of water per minute, from a depth of 40 feet? *Ans.* 6 horse-power.

6. How many tons of ore can be raised per hour from a mine 1800 feet deep, by an engine of 28 effective horse-power, reckoning 2,240 *lbs.* to the ton? *Ans.* $13\frac{3}{4}$ tons.

7. From what depth will an engine of 16 effective horse-power raise 5 *cwts.* of coal per minute? *Ans.* 943 feet, nearly.

8. In what time will an engine of 40 effective horse-power raise 44,000 cubic feet of water from a mine 360 feet deep, allowing $62\frac{1}{2}$ pounds to the cubic foot? *Ans.* 12 *h.* 30 *min.*

9. Required the quantity of work necessary to raise the material for a rectangular granite wall 25 feet long, $2\frac{1}{2}$ feet thick, and 20 feet high, the weight of granite being 162 *lbs.* per cubic foot?

Ans. $Q = 2{,}025{,}000$ *lb. ft.*

10. How long would it take an engine of 4 effective horse-power to raise the material for the wall in the last example?

Ans. $15\frac{1}{4}$ minutes, nearly.

11. What quantity of work must be expended in drawing a chain from a shaft, the length of the chain being 450 feet, and its weight 40 *lbs.* to the foot? *Ans.* 4,050,000 *lbs. ft.*

12. A cylindrical well is 150 feet deep, and 10 feet in diameter. Supposing the well to be filled with water to the depth of 50 feet, how much work must be expended in raising it to the top, water being taken at 62.5 *lbs.* per cubic foot? *Ans.* $Q = 30{,}679{,}687.5$ *lb. ft.*

13. What quantity of work will be required to overturn a right cone, with a circular base, whose altitude is 12 feet, and the radius of whose base is 4 feet, the weight of the material being estimated at 100 *lbs.* per cubic foot? *Ans.* $Q = 40{,}212.48$ *lb. ft.*

14. What is the kinetic energy of a body whose weight is 300 *lbs.* and whose velocity is 64 *ft.* per second? *Ans.* 1,910 *ft. lb.*, nearly.

15. What is the potential energy of a pond of water whose length is 100 *ft.*, breadth 50 *ft.*, and depth 3 *ft.*, and having a fall of 8 *ft.*, the weight of a cubic foot of water being $62\frac{1}{2}$ *lbs.*? *Ans.* 7,500,000 *lb. ft.*

16. A well 20 feet deep is full of water: what is the depth of water in the well when one quarter of the whole work required to empty it has been performed? *Ans.* 10 *ft.*

17. What is the effective horse-power of an engine that can pump 80 cubic feet of water per minute from a depth of 360 *ft.*, the weight of water being $62\frac{1}{2}$ *lbs.* per cubic foot? *Ans.* 54.5, nearly.

18. Show that the work required for overturning similar solids, similarly placed, varies as the fourth powers of their homologous lines.

Solution.—Denote the altitudes of the centres of gravity, by y and ry, the distances from the directions of the weights to the lines about which they turn, by x and rx, and their weights, by w and r^3w.

The quantity of work required to overturn the first, will be,

$$Q = w(\sqrt{x^2 + y^2} - y).$$

The quantity of work required to overturn the second, will be,

$$Q' = r^3w(\sqrt{r^2x^2 + r^2y^2} - ry) = r^4w(\sqrt{x^2 + y^2} - y).$$

Hence,

$$Q : Q' :: 1 : r^4 \quad . \quad . \quad Q.E.D.$$

Rotation.

154. When a body restrained by a fixed axis, about which it is free to turn, is acted upon by a force, it will, in general, take up a motion of **rotation** or **revolution.** In this kind of motion, each point of the body describes a circle, whose centre is in the axis, and whose plane is perpendicular to the axis. The time of a complete revolution being the same for each particle, it follows, that the velocities of the different particles will be proportional to their distances from the axis. The velocity of any particle will be equal to its distance from the axis multiplied by the angular velocity.

Quantity of Work of a Force Producing Rotation.

155. If a force is applied obliquely to the axis of rotation, we may conceive it to be resolved into two components, one parallel, and the other perpendicular, to the axis of rotation. The effect of the former will be counteracted by the resistance offered by the fixed axis; the effect of the latter in producing rotation will be exactly the same as that of the applied force. We need, therefore, only consider the component whose direction is perpendicular to the axis of rotation.

Let us suppose a body to be constrained to rotate around an axis through O, and perpendicular to the plane of the paper at O; let P represent a force acting in the plane of the paper; and let A and C be any two points of the body lying on the direction of P. Then will the plane of the paper be a plane of rotation through P.

Suppose the force P to turn the system through an infinitely small angle, and let B and D be the new positions of A

and C. Draw OE, Ba, and Dc respectively perpendicular to PE; draw also, AO, BO, CO, and DO. Denote the distances OA by r, OC by r', OE by p, and the path described by a point at a unit's distance from O, by θ'. Since the angles AOB, and COD are equal, from the nature of the motion of rotation, we shall have, $AB = r\theta'$, and $CD = r'\theta'$; and since the angular motion is infinitely small, these lines may be regarded as straight lines, perpendicular respectively to OA and OC. From the right-angled triangles ABa and CDc, we have,

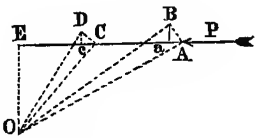

Fig. 125.

$$Aa = r\theta' \cos BAa, \quad \text{and} \quad Cc = r'\theta' \cos DCc.$$

In the right-angled triangles ABa, and OAE, we have AB perpendicular to OA, and Aa perpendicular to OE; hence, the angles BAa, and AOE, are equal, as are also their cosines; hence, we have,

$$\cos BAa = \cos AOE = \frac{p}{r}.$$

In like manner, it may be shown, that

$$\cos DCc = \cos COE = \frac{p}{r'}.$$

Substituting in the equations just deduced, we have,

$$Aa = p\theta', \text{ and } Cc = p\theta'; \quad \therefore \quad Aa = Cc;$$

whence,

$$P \cdot Aa = P \cdot Cc = Pp\theta'.$$

The first member of this equation is the quantity of work of P, when its point of application is at A; the second is the quantity of work of P, when its point of application is at C.

Hence, we conclude, that *the elementary quantity of work of a force applied to produce rotation, is always the same, wherever its point of application may be taken, provided its line of direction remains unchanged.*

We conclude, also, that the elementary quantity of work is equal to the intensity of the force *multiplied* by its lever arm and by the elementary space described by a point at a unit's distance from the axis.

If we suppose the force to act for a unit of time, the intensity and lever arm remaining the same, denoting the *angular velocity*, by θ, and the work by Q', we shall have,

$$Q' = Pp\theta.$$

For any number of forces similarly applied, the quantity of work being denoted by Q, we shall have,

$$Q = \Sigma (Pp)\theta \quad . \quad . \quad . \quad . \quad . \quad (254)$$

If the forces are in equilibrium, we shall have, $\Sigma (Pp) = 0$; consequently, $Q = 0$.

Hence, if any number of forces tending to produce rotation about a fixed axis, are in equilibrium, the entire quantity of work of the system of forces will be equal to 0.

Kinetic Energy of a Revolving Body.

156. Denote the angular velocity of a revolving body by θ, the masses of its elementary particles by m, m', etc., and their distances from the axis of rotation, by r, r', etc. Their velocities will be $r\theta$, $r'\theta$, etc., and their kinetic energies, $\frac{1}{2}mr^2\theta^2$, $\frac{1}{2}m'r'^2\theta^2$, etc. Denoting the entire kinetic energy of the body by L, we have, by summation, recollecting that θ is the same for all the terms,

$$L = \tfrac{1}{2}\Sigma (mr^2)\,\theta^2 \quad . \quad . \quad . \quad . \quad . \quad (255)$$

But $\Sigma (mr^2)$ is the moment of inertia of the body with re-

spect to the axis of rotation. Denoting the entire mass by M, and its radius of gyration, with respect to the axis of rotation, by k, we have,

$$\Sigma (mr^2) = Mk^2; \qquad \therefore\ L = \tfrac{1}{2}Mk^2\theta^2 \quad . \quad . \quad . \quad . \quad . \quad (256)$$

If, at any subsequent instant, the angular velocity has become θ', we have,

$$L' = \tfrac{1}{2}Mk^2\theta'^2;$$

and, for the gain or loss of kinetic energy in the interval,

$$L'' = \tfrac{1}{2}Mk^2(\theta'^2 - \theta^2) \quad . \quad . \quad . \quad . \quad . \quad (257)$$

If, in Equation (256), we make $\theta = 1$, we have,

$$L''' = \tfrac{1}{2}\Sigma (mr^2); \quad \text{or,} \quad \Sigma (mr^2) = 2L'''.$$

That is, the moment of inertia of a body, with respect to an axis, is equal to twice its kinetic energy when the angular velocity is equal to 1, or, to twice the quantity of work that must be expended to generate a unit of angular velocity.

The principle of kinetic energy, or living force, is applied in discussing the motion of machines. When the power performs more work than is necessary to overcome the resistances, the velocities of the parts increase, and a quantity of work is stored up, to be given out again when the resistances require more work to overcome them than is performed by the motor.

In many machines, pieces are introduced to equalize the motion; this is particularly the case when either the power or the resistance is variable. Such pieces are called *fly-wheels.*

Fly-Wheels.

157. A **fly-wheel** is a heavy wheel mounted on an axis, near the point of application of the force which it is designed

to regulate. It is generally composed of a rim, connected with an axis by radial arms. Sometimes it consists of radial bars, carrying spheres of metal at their outer extremities. Let us denote the mass of the wheel by M, its radius of gyration by k, the quantity of work stored up in any time by Q, and the initial and terminal angular velocities by θ' and θ''. We shall have, from Equation (257),

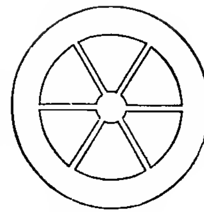

Fig. 126.

$$Q = \tfrac{1}{2}Mk^2(\theta''^2 - \theta'^2) \quad . \quad . \quad . \quad . \quad . \quad (258)$$

If $\theta'' > \theta'$, Q is positive and work is stored up; if $\theta'' < \theta'$, Q is negative, and the wheel gives out work.

If the angular velocity increase from θ' to θ'', and then decrease to θ', and so on, alternately, the work accumulated during the first part of each cycle is given out during the second part, and any device that will make θ' and θ'' more nearly equal, will contribute toward equalizing the motion of the machine. By suitably increasing the mass and radius of gyration, their difference may be made as small as desirable. Let the half-sum of the greatest and least angular velocities be called **the mean angular velocity**, and denote it by θ'''. We shall have $\dfrac{\theta'' + \theta'}{2} = \theta'''$, and by factoring the second member of (258), we have,

$$Q = \tfrac{1}{2}Mk^2(\theta'' + \theta')(\theta'' - \theta');$$

whence, by substituting the value of $\theta'' + \theta'$,

$$Q = Mk^2(\theta'' - \theta')\,\theta''' \quad . \quad . \quad . \quad . \quad . \quad (259)$$

Let us suppose the difference between the greatest and

least velocity, equal to the n^{th} part of their mean, that is, that

$$\theta'' - \theta' = \frac{\theta'''}{n}.$$

This, in (259), gives

$$Q = \frac{Mk^2\theta'''^2}{n}; \quad \text{or,} \quad Mk^2 = \frac{nQ}{\theta'''^2}. \quad . \quad . \quad . \quad . \quad (260)$$

From this equation the moment of inertia of the wheel may be found, when we know n, Q, and θ'''. The value of n may be assumed; for most kinds of work a value of from 6 to 10 will be found to give sufficient uniformity; the value of θ''' depends on the character of the work to be performed, and Q is made known by the character of the motion to be regulated.

To find the value of Q in any practical case we must find the quantity of work that is *stored up* in the first part of the cycle, which is also the quantity *given out* in the second part of the cycle. The method of proceeding is best illustrated by a practical example.

The Crank and Crank Motion.

158. A **crank** is a device for converting *reciprocating motion* into *rotary motion,* or the reverse.

In the diagram, DC represents a **reciprocating piece,** that is, a piece that moves backward and forward being restrained by suitable guides, as is the case with the *piston rod* in a locomotive; CB is a **connecting rod** having a hinge joint at C and connected with AB by means of a short axle called the **crank pin**; AB is the **crank arm** solidly connected with an axle which is perpendicular to the plane CBA, and which is called the **crank axis.**

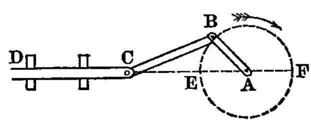

Fig. 127.

When the crank pin is at E the reciprocating piece is at one limit of its play, and when the crank pin is at F the reciprocating piece is at the other limit of its play. The distance EF, which is equal to twice the length of the crank arm, is called the **throw of the crank**; it measures the distance through which the reciprocating motion of CD takes place.

If we suppose a force P to act along DC, either pushing or pulling, it will have no tendency to produce rotation of the crank when the crank pin is either at E or F; these points are therefore called **dead points.** When the crank pin is at any other point, P will tend to produce rotation, the measure of the tendency being equal to the component of P in the direction of BC, *multiplied by* the perpendicular distance from A to this direction.

If we suppose P to be constant, its component in the direction of BC will be variable, and it is obvious that the variation will increase with the obliquity of BC to CD. For this reason the connecting rod should be made as long as the nature of the mechanism will permit; when it is 10 or 12 times as long as the crank arm, we may, in most practical cases, regard the component along BC as constant, and as acting parallel to CD. We shall so regard it in the following discussion.

We may suppose the force P to act in one direction, say in the direction from C to D; or we may suppose it to act alternately in the directions from C to D and from D to C. In the former case the crank is said to be a **crank of single action,** and in the latter case it is a **crank of double action.**

We have an example of the *former* in the ordinary lathe where the force applied to the treadle only acts downward; we have an example of the *latter* in the locomotive where the force of the steam acts alternately in both directions.

Fly-Wheels to Regulate Crank Motion.

159. To compute the dimensions of a fly-wheel to regulate crank motion, let us *first* consider the case of a crank of single action.

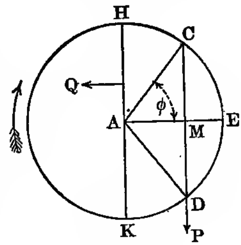

Fig. 128.

Let us take the connecting rod so long that it may, in all positions, be regarded as sensibly parallel to the reciprocating piece, which is supposed to lie in the direction HK, and suppose the force P to act downward only, and through a distance equal to HK, the diameter of the circle HCK described by the crank pin. Suppose the resistance overcome to be constant and equal in effect to a force Q acting with a lever arm 1. Denote the length of the crank arm by r.

The work of P in one revolution is equal to $P \times 2r$, and the work of Q in the same period is $Q \times 2\pi$; these are, by hypothesis, equal to each other, that is,

$$2\pi Q = 2Pr, \quad \text{or} \quad \frac{Q}{Pr} = \frac{1}{\pi} \quad . \quad . \quad . \quad . \quad . \quad (261)$$

When the crank pin is at H, the moment of P is 0; as the crank pin advances in its path, HCD, the moment of P increases, being equal to $P \times AM$. As long as $P \times AM <$ $Q \times 1$, the work of the power is *less than* the work of the resistance; when $P \times AM = Q \times 1$, the work of the power is *equal to* the work of the resistance; and when $P \times M >$ $Q \times 1$, the work of the power is *greater than* the work of the resistance. In the last case the excess of work of the power is being stored up in the fly-wheel which is supposed to be mounted on the crank axis.

To find the position of the crank arm when work begins to be stored up, make

$$P \times AM = Q \times 1, \quad \text{whence} \quad \frac{AM}{r} = \frac{Q}{Pr} \quad . \; . \; . \; . \; . \quad (262)$$

Denoting the corresponding angle EAC by ϕ, the first member of (262) is equal to $\cos \phi$, and from (261) the second member is the reciprocal of π; hence, we have,

$$\cos \phi = \frac{1}{\pi}, \quad \text{or} \quad \phi = 71^\circ\ 27' \quad . \; . \; . \; . \; . \quad (263)$$

Whilst the crank pin is moving over the arc CED, equal to $142^\circ\ 54'$, work is being stored up. To find the amount of work stored up, we observe that the corresponding work of P is $P \times CD$, or to $P \times 2r \sin 71^\circ\ 27'$, and denoting this work by W, we have,

$$W = P \times 2r \sin 71^\circ 27' = 2Pr \times 0.9480 \quad . \; . \; . \; . \; . \quad (264)$$

The work of Q in the same period, denoted by W', is given by the equation,

$$W' = Q \times 2\pi \times \frac{142^\circ\ 54'}{360^\circ}.$$

Or, because, $Q \times 2\pi = P \times 2r$, we have,

$$W' = 2Pr \times \frac{142^\circ\ 54'}{360^\circ} = 2Pr \times 0.3968 \quad . \; . \; . \; . \; . \quad (265)$$

Subtracting (265) from (264), we have, for the amount of work stored up,

$$W - W' = 2Pr\,[.9480 - .3968] = Pr \times 1.1024 \quad . \; . \; . \; . \quad (266)$$

Substituting this for Q, in Equation (260), we have,

$$Mk^2 = \frac{n \times Pr \times 1.1024}{\theta'''^2} \quad . \; . \; . \; . \; . \quad (267)$$

If we denote the number of revolutions of the crank per second by N, the value of θ''' will be $2\pi N$; this reduces (267) to the form,

$$Mk^2 = \frac{n \times Pr \times 1.1024}{4\pi^2 N^2} \quad . \; . \; . \; . \; . \quad (268)$$

Poncelet says that a value of n between 7.5 and 10 will secure a sufficient degree of regularity. Making $n = 10$, and substituting for π^2 its value 9.87, we have finally,

$$Mk^2 = \frac{Pr \times 11.024}{39.48\, N^2} \quad . \; . \; . \; . \; . \quad (269)$$

The first member of (269) is the moment of inertia of the required fly-wheel, and the second member is its value; this latter is completely known from the conditions of the problem.

We may now assume k, the radius of gyration, and compute M; or, we may assume M, or, what is the same thing, we may assume the weight of the wheel, and compute k. In either case the problem is completely solved.

Crank of Double Action.

160. Let us next consider a crank of double action, employing the same notation as before. Let the force P act downward through the distance, $2r$, and then upward through same distance; the quantity of work of P in an entire revolution of the crank will then be equal to $4Pr$, and we shall have, as before,

$$4Pr = 2\pi Q, \quad \text{or} \quad \frac{Q}{Pr} = \frac{2}{\pi} \; . \; . \; . \; (270)$$

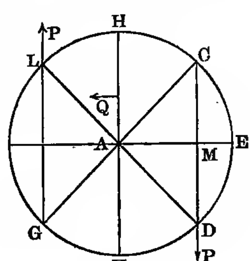

Fig. 129.

When the moment of P is equal to the moment of Q, we have,

$$P \times AM = Q, \quad \text{or} \quad \frac{AM}{r} = \frac{Q}{Pr}.$$

Whence, by substitution, as before, we have,

$$\cos\phi = \frac{2}{\pi}, \quad \text{or} \quad \phi = 50°\,28' \quad \ldots\ldots \quad (271)$$

There will be four points, C, D, G, and L, at which there will be an equilibrium between P and Q. In passing from C to D the velocity will increase; from D to G the velocity will decrease; from G to L the velocity will increase again; and from L to C the velocity will decrease. From C to D work will be stored up to be given in passing from D to G, and in passing from G to L an equal amount of work will be stored, to be given out in passing from H to C; that is, there are two cycles of change in each revolution.

The work stored up in passing from C to D is given by the equation,

$$W - W' = P \times 2r\sin 50°\,28' - Q \times 2\pi\frac{100°\,56'}{360°}.$$

Or, by reduction,

$$W - W' = Pr(1.54 - 1.12) = Pr \times 0.42.$$

Substituting this for Q in (260), and making $n = 10$, we have,

$$Mk^2 = \frac{Pr \times 4.2}{\theta'''^2} \quad \ldots\ldots \quad (272)$$

Denoting the number of revolutions per second by N, the value of θ''' is $2\pi N$, which reduces (272) to

$$Mk^2 = \frac{Pr \times 4.2}{39.48\,N^2}, \quad \ldots\ldots \quad (273)$$

which gives the moment of inertia of the required fly-wheel.

To compare the results in (269) and (273), let us suppose the work of the power during one revolution to be the same in both. In this case the value of P in (273) is one half that

of P in (269); consequently the moment of inertia of the fly-wheel to regulate single crank action is $11.024 \div 2.1$, or $5\frac{1}{4}$ times that of the fly-wheel to regulate double crank action.

Elasticity.—Impact.

161. Elasticity is that property by virtue of which a body tends to resume its original form after having been distorted by the action of some force. A body may be distorted *by compression, by extension,* or *by twisting.* The force that acts to change the form of a body may be called the **force of distortion,** and the force that tends to restore the body to its original shape may be called the **force of restitution.** If the force of restitution is equal to the force of distortion, the body is **perfectly elastic**; if the force of restitution is 0, the body is **perfectly inelastic.** No body is either perfectly elastic or perfectly inelastic. Ivory, glass, and steel are among the more elastic bodies; soft putty and tempered clay are examples of bodies that are nearly inelastic.

If we denote the force of distortion by d, the force of restitution by r, and their ratio by e, we have,

$$e = \frac{r}{d} \quad . \quad . \quad . \quad . \quad . \quad (274)$$

The value of e, which depends upon the molecular action of the particles of a body, can be taken as a measure of the body's elasticity; it may then be called **the coefficient of elasticity.**

If one ball impinges upon another a succession of effects takes place which are dependent upon the molecular constitution of the two bodies, and in general these effects take place in an exceedingly short space of time, so that the force exerted may be called **impulsive.** The first effect of the impact is to produce a compression of both bodies, and this

continues until the molecular forces called into action are sufficient to resist further distortion. The balls, being endowed with a certain degree of elasticity, then tend to recover their spherical form, and in this effort they exert a further pressure upon each other, the force of restitution being equal to e times the force of distortion, (Eq. 274).

Momentum and Velocity.

162. In the case of impact of two spherical balls, A and B, let us denote the mass of A by m and the mass of B by m', and furthermore let us denote the velocity of A before impact by v and that of B by v', v being greater than v'. Before collision the aggregate momentum of the two bodies will be equal to $mv + m'v'$. When collision takes place a mutual pressure P will be set up, the effect of which will be to diminish the velocity of A and to increase that of B, and because action and reaction are equal, the momentum of A will be diminished as much as that of B is increased, and the same law will continue during the period of restitution. Hence, the *aggregate momentum* is not changed during the period of impact.

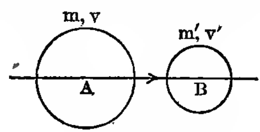

Fig. 130.

Let us assume, as shown in the figure, that the centres of the balls are moving along the same right line, the velocity of A being v and that of B being v'. At any time t, before or after impact, denote the distance of A from a fixed plane, at right angles to the given direction, by x, and that of B from the same plane by x': the distance of the centre of gravity of A and B from the same plane will be given by the equation,

$$x_1 = \frac{mx + m'x'}{m + m'} \quad . \quad . \quad . \quad . \quad . \quad (275)$$

At the time $t + dt$ the distance of A from the fixed plane

will be $x + vdt$ and the distance of B from the same plane will be $x' + v'dt$, and the distance of the centre of gravity of A and B from that plane will be given by the equation,

$$x_1 + dx_1 = \frac{m(x + vdt) + m'(x' + v'dt)}{m + m'} \quad . \quad . \quad . \quad . \quad (276)$$

Subtracting (275) from (276), and dividing by dt, we have,

$$\frac{dx_1}{dt} = \frac{mv + m'v'}{m + m'} \quad . \quad . \quad . \quad . \quad . \quad (277)$$

It has just been shown that the second member of (277), which represents the aggregate momentum of the two bodies, is unchanged by the collision; hence, the first member, which represents the velocity of the centre of gravity, is also unchanged by the collision.

Direct Central Impact.

163. If the centres of the two balls (Fig. 130) are moving along the same straight line when the collision takes place, the impact is said to be **direct** and **central**. Let us consider the direct and central impact of two spheres of the same material, and let e denote their common coefficient of elasticity.

Denote the mass of the ball A by m, its velocity before impact by v, and its velocity after impact by u; denote the mass of the ball B by m', its velocity before impact by v', and its velocity after impact by u'; also, denote the common velocity of the two balls at the instant of greatest compression by w.

The aggregate momentum, or simply the momentum of the two bodies before impact, is $mv + m'v'$, and their momentum at the instant of maximum compression is $(m + m')w$; but from Art. 162, these two expressions are equal; hence,

$$(m + m')w = mv + m'v', \quad \text{or} \quad w = \frac{mv + m'v'}{m + m'} \quad . \quad . \quad . \quad . \quad (278)$$

To find the velocities lost by A and gained by B during *the period of compression,* we subtract w from v and v' from w, giving after reduction,

$$v - w = \frac{m'}{m + m'}(v - v'), \quad . \quad . \quad . \quad . \quad . \quad (279)$$

$$w - v' = \frac{m}{m + m'}(v - v'). \quad . \quad . \quad . \quad . \quad . \quad (280)$$

The velocities lost by A and gained by B during *the period of restitution* will be equal respectively to the second members of (279) and (280), each multiplied by e (Art. 161): hence, the total loss of velocity by A, and gain of velocity by B during the entire impact, will be given by the equation,

$$v - u = (1 + e)\frac{m'}{m + m'}(v - v'), \quad . \quad . \quad . \quad . \quad . \quad (281)$$

$$u - v' = (1 + e)\frac{m}{m + m'}(v - v'). \quad . \quad . \quad . \quad . \quad . \quad (282)$$

The velocities of A and B after collision, found from the preceding equations by simple transposition, are

$$u = v - (1 + e)\frac{m'}{m + m'}(v - v'), \quad . \quad . \quad . \quad . \quad . \quad (283)$$

$$u' = v' + (1 + e)\frac{m}{m + m'}(v - v'). \quad . \quad . \quad . \quad . \quad . \quad (284)$$

We have supposed both bodies to move in the same direction; if we suppose them to move in opposite directions before impact, we must make $v' = -v'$ in the preceding equations.

Discussion.

1°. Let the bodies be equal in mass and suppose them to be perfectly elastic: we then have $\frac{m'}{m+m'}$ and $\frac{m}{m+m'}$ each equal to $\frac{1}{2}$, and $1 + e = 2$, which in (283) and (284) give, after reduction,

$$u = v', \quad \text{and} \quad u' = v. \quad . \quad . \quad . \quad . \quad . \quad (285)$$

That is, *the bodies after impact will move in the same direction, having exchanged velocities.*

2°. The same supposition as before, except that the motion of B is reversed, that is, $v' = -v'$, gives

$$u = -v', \quad \text{and} \quad u' = v. \quad . \quad . \quad . \quad . \quad . \quad (286)$$

That is, *the bodies recoil, having exchanged velocities.*

3°. If we suppose B to be at rest before collision, that is, $v' = 0$, both bodies being perfectly elastic, we have

$$u = 0, \quad \text{and} \quad u' = v, \quad . \quad . \quad . \quad . \quad . \quad (287)$$

which is only a particular case of the previous supposition. *The first body comes to rest and the second one moves on with its velocity.*

4°. Let both bodies be perfectly inelastic, in which case $e = 0$. This supposition reduces (283) and (284) to

$$u = v - \frac{m'}{m+m'}(v - v') = \frac{mv + m'v'}{m+m'}, \quad . \quad . \quad . \quad . \quad . \quad (288)$$

$$u' = v' + \frac{m}{m+m'}(v - v') = \frac{mv + m'v'}{m+m'}. \quad . \quad . \quad . \quad . \quad . \quad (289)$$

That is, *the bodies after impact will move on with equal velocities.*

5°. Let both bodies be perfectly elastic, and suppose the mass of the second body to be infinite, and also to be at rest. In this case, the quantity

$$\frac{m'}{m+m'} = \frac{1}{1+\frac{m}{m'}} = 1, \quad \text{and} \quad \frac{m}{m+m'} = 0;$$

whence we have,

$$u = v - 2v = -v, \quad \text{and} \quad u' = 0. \; . \; . \; . \; . \; . \quad (290)$$

That is, *the first body will recoil, having reversed its velocity.*

Loss of Kinetic Energy by Impact.

164. The kinetic energy of the two balls considered will be $\frac{1}{2}(mv^2+m'v'^2)$ before collision and it will be $\frac{1}{2}(mu^2+m'u'^2)$ after collision; the loss of kinetic energy in consequence of collision, denoted by L, will therefore be,

$$L = \tfrac{1}{2}m(v^2-u^2) + \tfrac{1}{2}m'(v'^2-u'^2). \; . \; . \; . \; . \; . \quad (291)$$

Substituting the values of u and u' from (283) and (284), we have,

$$L = \tfrac{1}{2}m\left[2(1+e)\frac{m'}{m+m'}(v-v')\,v - (1+e)^2\frac{m'^2}{(m+m')^2}(v-v')^2\right]$$

$$-\tfrac{1}{2}m'\left[2(1+e)\frac{m}{m+m'}(v-v')\,v' + (1+e)^2\frac{m^2}{(m+m')^2}(v-v')^2\right]$$

or,

$$2L = \frac{2mm'}{m+m'}(1+e)(v-v')^2 - \frac{mm'(m+m')}{(m+m')^2}(1+e)^2(v-v')^2,$$

or, $$2L = \frac{mm'}{m+m'}(1+e)(v-v')^2[2-(1+e)];$$

or, finally,

$$L = \tfrac{1}{2}(1-e^2)(v-v')^2\frac{mm'}{m+m'}. \quad . \quad . \quad . \quad . \quad . \quad (292)$$

If the bodies are perfectly elastic, $e = 1$, and (292) becomes

$$L = 0. \quad . \quad . \quad . \quad . \quad . \quad (293)$$

That is, *there is no loss of kinetic energy.*

If the bodies are perfectly inelastic, $e = 0$, and (292) reduces to

$$L = \frac{1}{2}\frac{mm'}{m+m'}(v-v')^2, \quad . \quad . \quad . \quad . \quad . \quad (294)$$

which is the maximum loss of kinetic energy.

For intermediate values of e the loss of kinetic energy will lie between the values of L already deduced, and the loss in any case can be computed when e is known.

The preceding discussion shows that in machines where shocks are inevitable, the colliding pieces should be made of as elastic material as possible.

Examples.

1. Two perfectly elastic bodies whose weights are respectively 10 *lbs.* and 16 *lbs.* collide with velocities which are respectively 12 *ft.*, and -6 *ft.* Find the results of the impact.

Ans. The first body will recoil with a velocity equal to -10.154 *ft.*, and the second body will also retrace its path with a velocity equal to 7.846 *ft.* The kinetic energy will be unchanged.

2. If, in the preceding problem, we suppose both bodies to be perfectly inelastic, what are the results of the impact?

Ans. The balls will move together with a common velocity equal to 0.92. The loss of kinetic energy will be equal to 997 *ft. lbs.*, nearly.

3. With what velocity must an inelastic body weighing 8 *lbs.* impinge upon an inelastic body at rest and weighing 25 *lbs.* to impart to it a velocity of 2 *ft.* per second? *Ans.* $v = 8\frac{1}{4}$ *ft.* per second.

4. If both bodies are supposed perfectly elastic in the preceding problem, what is the required velocity? *Ans.* $v = 4\frac{1}{8}$ *ft.* per second.

Oblique Impact.

165. Let the ball A, whose mass is m and whose velocity in the direction EA is v, collide with the ball B, whose mass is m' and whose velocity in the direction $E'B$ is v' at the point C. After impact suppose that A moves in the direction AD with a velocity u, and that B moves in the direction BD' with the velocity u'.

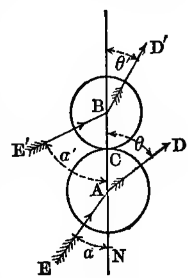

Fig. 131.

Let AB be the common normal to the two bodies at C, and denote the angles between it and EA, $E'B$, AD, and BD' respectively by α, α', θ, and θ'. Before impact the velocities in the direction of the normal are $v \cos \alpha$ and $v' \cos \alpha'$ and the velocities in the direction of the tangent at C are $v \sin \alpha$ and $v' \sin \alpha'$; after impact the velocities in the direction of the normal are $u \cos \theta$ and $u' \cos \theta$ and the velocities in the direction of the tangent are $u \sin \theta$ and $u' \sin \theta'$. It is obvious that the velocities in the direction of the tangent are totally unaffected by the impact, whilst those in the direction of the normal are affected as though the others did not exist. Hence, the relations between the

velocities in the direction of the normal before and after impact can be obtained at once from (283) and (284) by simply replacing v, v', u, and u' by the corresponding normal components; hence,

$$u\cos\theta = v\cos\alpha - (1+e)\frac{m'}{m+m'}(v\cos\alpha - v'\cos\alpha') \quad . . (295)$$

$$u'\cos\theta' = v'\cos\alpha' + (1+e)\frac{m}{m+m'}(v\cos\alpha - v'\cos\alpha') \quad . . (296)$$

To find the direction of AD, the path of A after impact, we see that A is moving in the direction of the tangent with a velocity $v\sin\alpha$ and in the direction of the normal with a velocity $u\cos\theta$; hence, from the figure, we have,

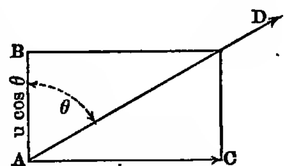

Fig. 132.

$$\cot\theta = \frac{u\cos\theta}{v\sin\alpha}.$$

Substituting for $u\cos\theta$ its value from (295) and reducing, we have,

$$\cot\theta = \cot\alpha - (1+e)\frac{m'}{m+m'}\left(\cot\alpha - \frac{v'\cos\alpha'}{v\sin\alpha}\right) \quad . . . . (297)$$

and in like manner,

$$\cot\theta' = \cot\alpha' + (1+e)\frac{m}{m+m'}\left(\frac{v\cos\alpha}{v'\sin\alpha'} - \cot\alpha'\right) \quad . . . . (298)$$

Equations (295), (296), (297), and (298) make known all the circumstances of motion after impact.

As a single illustration, let us suppose that both A and B are perfectly elastic, but that B has an infinite mass and is at

rest. We then have $m' = \infty$ and $v' = 0$; substituting in (297), observing that $\frac{m'}{m + m'} = 1$, and $1 + e = 2$, we have,

$$\cot \theta = -\cot \alpha, \quad \text{or} \quad \theta = 180^\circ - \alpha.$$

Remembering the directions in which θ and α are estimated, we see that the ball A recoils, making the *angle of reflection* equal to the *angle of incidence.*

VIII.—MECHANICS OF LIQUIDS.

Classification of Fluids.

166. A **fluid** is a body whose particles move freely amongst each other, each particle yielding to the slightest force. Fluids are of two classes: **liquids**, of which water is a type, and **gases**, or **vapors**, of which air and steam are types. The distinctive property of the first class is, that they are sensibly **incompressible**; thus, water, on being pressed by a force of 15 *lbs.* on each square inch of surface, only suffers a diminution of about $\frac{1}{200,000}$ of its bulk. The second class comprises those which are **readily compressible**; thus, air and steam are easily compressed into smaller volumes, and when the pressure is removed, they expand, so as to occupy larger volumes.

Most liquids are imperfect; that is, there is more or less adherence between their particles, giving rise to **viscosity**. In what follows, they will be regarded as **destitute of viscosity**, and **homogeneous**. For certain purposes, fluids may also be regarded as destitute of weight, without impairing the validity of the conclusions.

Principle of Equal Pressures.

167. From the nature and constitution of a fluid, it follows that each of its particles is perfectly movable in all directions. From this fact, we deduce the following fundamental law, viz.: *If a fluid is in equilibrium under the action of any forces whatever, each particle of the mass is equally pressed in all directions*; for, if any particle were more strongly pressed in one direction than in the

others, it would yield in that direction, and motion would ensue, which is contrary to the hypothesis.

This is called the **principle of equal pressures.**

It follows, from the principle of equal pressures, that if any point of a fluid in equilibrium be pressed by any force, that pressure will be transmitted without change of intensity to every other point of the fluid mass.

This may be illustrated experimentally, as follows:

Let AB represent a vessel filled with a fluid in equilibrium. Let C and D represent two openings, furnished with tightly-fitting pistons. Suppose that forces are applied to the pistons just sufficient to maintain the fluid mass in equilibrium. If, now, any additional force be applied to the piston P, the piston Q will be forced outward; and in order to prevent this, and restore the equilibrium, it will be found necessary to apply a force to the piston Q, which shall have the same ratio to the force applied at P that the area of the piston Q has to the area of the piston P. This principle will be found to hold true, whatever may be the sizes of the two pistons, or in whatever portions of the surface they may be inserted. If the area of P be taken as a unit, then will the pressure upon Q be equal to the pressure on P, multiplied by the area of Q.

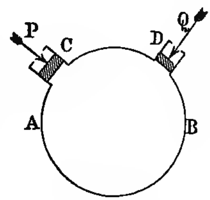

Fig. 133.

The pressure transmitted through a fluid in equilibrium, to the surface of the containing vessel, is *normal* to that surface; for if it were not, we might resolve it into two components, one normal to the surface, and the other tangential; the effect of the former would be destroyed by the resistance of the vessel, whilst the latter would impart motion to the fluid, which is contrary to the supposition of equilibrium.

In like manner, it may be shown, that the resultant of all

the pressures, acting at any point of the free surface of a fluid, is normal to the surface at that point. When the only force acting is the force of gravity, the surface is **level.**

For comparatively small areas, a level surface coincides sensibly with a horizontal plane. For larger areas, as lakes and oceans, a level surface coincides with the general surface of the earth. Were the earth at rest, the level surface of lakes and oceans would be spherical; but, on account of the centrifugal force arising from the rotation of the earth, it is an ellipsoidal surface, whose axis of revolution is the axis of the earth.

Pressure Due to Weight.

168. If an incompressible fluid be in a state of equilibrium, the pressure at any point of the mass arising from the weight of the fluid, is proportional to the depth of the point below the free surface.

Take an infinitely small surface, supposed horizontal, and conceive it to be the base of a vertical prism whose altitude is equal to its distance below the free surface. Conceive this filament to be divided by horizontal planes into infinitely small, or elementary prisms. It is evident, from *the principle of equal pressures*, that the pressure upon the lower face of any one of these elementary prisms is greater than that upon its upper face, by the weight of the element, whilst the lateral pressures are such as to counteract each other's effects. The pressure upon the lower face of the first prism, counting from the top, is, then, just equal to its weight; that upon the lower face of the second is equal to the weight of the first, *plus* the weight of the second, and so on to the bottom. Hence, the pressure upon the assumed surface is equal to the weight of the entire column of fluid above it.

Had the assumed elementary surface been oblique to the horizon, or perpendicular to it, and at the same depth as be-

fore, the pressure upon it would have been the same, from *the principle of equal pressures.* We have, therefore, the following law:

The pressure upon any elementary portion of the surface of a vessel containing a heavy fluid is equal to the weight of a prism of the fluid whose base is equal to that surface, and whose altitude is equal to its depth below the free surface.

Denoting the area of the elementary surface by dm, its depth below the free surface by z, the weight of a unit of volume of the fluid by w, and the pressure by p, we have,

$$p = wzdm \quad . \quad . \quad . \quad . \quad . \quad (299)$$

To find the pressure on the interior surface of a vessel, or on any submerged surface due to the weight of the liquid. Conceive the surface to be divided into infinitesimal elements, and let any one of these elements be denoted by dm, and let its distance from the free surface of the fluid be denoted by z. Then from what precedes, we have, for the pressure on this element,

$$p = wzdm,$$

and like expressions for the pressures on the other elements. These pressures are everywhere normal to the surface, and their aggregate may be called the **bursting pressure.** This aggregate cannot, in general, be found by the process of integration, but if we denote it by P, we shall always have,

$$P = \Sigma\,(wzdm) = w\Sigma\,(zdm), \quad . \quad . \quad . \quad . \quad . \quad (300)$$

the sign of summation being extended to include every element of the surface.

If we regard the surface pressed as material, we have, from

equation (30), by changing the sign of *integration* to that of *summation*, and solving,

$$\Sigma(zdm) = z_1 \Sigma(dm).$$

Substituting, in (300), we have,

$$P = wz_1 \Sigma(dm) \quad . \quad . \quad . \quad . \quad . \quad (301)$$

In (301), P is the pressure on the interior surface of the vessel, z_1 is the distance of the centre of gravity of the surface pressed below the free surface, and $\Sigma(dm)$ is the area of the surface pressed; hence,

The entire pressure of a heavy fluid on the interior of the containing vessel, is equal to the weight of a volume of the fluid, whose base is equal to the area of the surface pressed, and whose altitude is equal to the distance of the centre of gravity of the surface from the free surface of the fluid.

Let it be required to find the *vertical component* of the pressure on any element. We have seen that the pressure on any element of a surface is normal to the surface. Denote the angle which this normal makes with the vertical, estimated from above downward, by ϕ, and resolve the pressure into two components, one vertical and the other horizontal. Denoting the vertical component by p', we have,

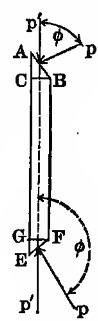

Fig. 134.

$$p' = p \cos\phi = wzdm \cos\phi.$$

But $dm \cos\phi$ is equal to the horizontal projection of dm, in other words it is a horizontal section of a vertical prism of which dm is the base; denoting this cross section by dm', we have,

$$p' = wzdm' \quad . \quad . \quad . \quad . \quad . \quad (302)$$

If $\phi < 90°$, p' will be *positive*, and the pressure will be downward; if $\phi > 90°$, p' will be negative, and the pressure will be upward.

Let figure 134 represent a submerged vertical filament: from what precedes, we see that it will be pressed downward by a force that is equal to the weight of a column of the fluid whose base is the cross-section of the filament and whose altitude is the distance of AB from the free surface; and it will be pressed upward by a force that is equal to the weight of a column of the fluid whose base is the cross-section of the filament and whose altitude is the distance of EF from the free surface.

This principle is used hereafter in finding an expression for the buoyant effect of a fluid.

EXAMPLES.

1. A hollow sphere is filled with a liquid. How does the entire pressure, on the interior surface, compare with the weight of the liquid?

Ans. The pressure is three times the weight of the liquid.

2. A hollow cylinder, with a circular base, is filled with a liquid. How does the pressure on the interior surface compare with the weight of the liquid?

Ans. The pressure is equal to $\frac{r+h}{r}$ times the weight of the liquid; here h is the height of the cylinder, and r is the radius of its base.

3. A right cone, with a circular base, stands on its base, and is filled with a liquid. How does the pressure on the internal surface compare with the weight of the liquid?

Ans. The pressure equals $\frac{3r + 2\sqrt{h^2 + r^2}}{r}$ times the weight of the liquid; here h is the altitude, and r the radius of the base.

4. Required the relation between the pressure and the weight in the preceding case, when the cone stands on its vertex.

Ans. The pressure is $\frac{\sqrt{h^2 + r^2}}{r}$ times the weight of the liquid.

5. What is the pressure on the lateral faces of a cubical vessel filled with water, the edges of the cube being 4 feet, and the weight of the water $62\frac{1}{2}$ *lbs.* per cubic foot? *Ans.* 8,000 *lbs.*

6. A cylindrical vessel is filled with water. The height of the vessel is 4 feet, and the radius of the base 6 feet. What is the pressure on the lateral surface? *Ans.* 18,850 *lbs.*, nearly.

7. A sluice-gate 10 feet square is placed vertically in the water, its upper side coinciding with the free surface; what are the respective pressures on the upper and lower halves, the weight of water being $62\frac{1}{2}$ *lbs.* per cubic foot? *Ans.* 7,812.5 *lbs.* and 23,437.5 *lbs.*

8. What are the pressures on the two triangles formed by drawing one diagonal of the gate? *Ans.* $10,416\frac{2}{3}$ *lbs.* and $20,833\frac{1}{3}$ *lbs.*

Centre of Pressure.

169. The **centre of pressure** on a surface is the point at which the resultant pressure intersects the surface.

Let us consider the case of a submerged plane surface, $ABED$, whose prolongation intersects the free surface, XOF, of the liquid in OX, and whose inclination to the free surface is equal to α. Let OX be taken as the axis of X, and let OY, perpendicular to OX, and lying in the plane of $ABED$, be taken as the axis of Y, the value of y being positive downward.

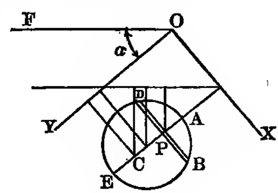

Fig. 135.

Let $ABED$ be divided into infinitesimal elements by lines parallel to OX; let DB, denoted by dm, be one of these elements, and denote the co-ordinates of its centre of gravity, P, by x and y.

The resultant pressure on DB will pass through P, and because the distance of P below the free surface is equal to $y \sin \alpha$, the value of this pressure will be, from (299),

$$w \cdot y \sin \alpha \cdot dm,$$

and because it is perpendicular to $ABED$, its moment with respect to the axis of X will be

$$wy^2 \sin\alpha dm,$$

and the sum of the moments of all of the elements with respect to OX will be

$$w \sin\alpha \int y^2 dm \quad . \quad . \quad . \quad . \quad . \quad (303)$$

Denote the co-ordinates of the centre of gravity of $ABED$ by x_1 and y_1, and the co-ordinates of the centre of pressure, C, by x' and y'. Then, from Art. 168, the entire pressure on $ABED$ will be

$$wy_1 \sin\alpha \int dm,$$

and its moment with respect to OX will be

$$y' \times w \sin\alpha y_1 \int dm \quad . \quad . \quad . \quad . \quad . \quad (304)$$

But the expressions (303) and (304) are equal; equating them, and finding the value of y', we have,

$$y' = \frac{\int y^2 dm}{y_1 \int dm} \quad . \quad . \quad . \quad . \quad . \quad (305)$$

In like manner, the sum of the moments of the elementary pressures with respect to the axis of Y is

$$w \sin\alpha \int xydm,$$

and that of the entire pressure with respect to the same axis is

$$x'w \sin\alpha x_1 \int dm.$$

Equating these expressions, and solving, we have,

$$x' = \frac{\int xydm}{x_1 \int dm} \quad . \quad . \quad . \quad . \quad . \quad (306)$$

If the surface has a line of symmetry perpendicular to OX, which is generally the case in practice, we may take that line as the axis of Y, in which case x' will be equal to 0, and the centre of pressure will be determined by equation (305).

Equation (305) shows that *the distance of the centre of pressure on the plane surface from the intersection of its plane with the free surface, is equal to the moment of inertia of the surface divided by the moment of the area of the surface, both taken with respect to the line of intersection.*

If we denote the moment of inertia of the surface by $M(k^2 + y_1^2)$, the moment of the mass will be equal to My_1, and from equation (305), we have,

$$y' = \frac{k^2}{y_1} + y_1, \quad . \quad . \quad . \quad . \quad . \quad (307)$$

in which k is the principal radius of gyration parallel to OY, and y_1 is the distance of the centre of gravity from the same line OX.

Because equation (305) is entirely independent of α, the position of the centre of pressure with respect to OX is independent of the inclination of the plane.

EXAMPLES.

1. Where is the centre of pressure on a rectangular flood-gate, the upper line of the gate coinciding with the surface of the water.

SOLUTION.—If we denote the length of the gate by $2l$, we have from (217), $k^2 = \frac{1}{3}l^2$; we also have $y_1 = l$. Substituting them in (307), we have,

$$y' = \tfrac{4}{3}l = \tfrac{2}{3}(2l).$$

Hence, the centre of pressure is on the line of symmetry and two thirds of the distance from the top to the bottom of the gate.

2. Let it be required to find the pressure on a submerged rectangular

flood-gate $ABCD$, the plane of the gate being vertical. Also, the distance of the centre of pressure below the surface of the water.

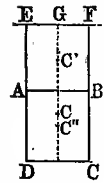

Fig. 136.

SOLUTION.—Let EF be the intersection of the plane with the surface of the water, and suppose the rectangle AC to be prolonged till it reaches EF. Let C, C', and C'', be the centres of pressure of the rectangles EC, EB, and AC, respectively. Denote the distance GC'', by z, the distance ED, by a, and the distance EA, by a'. Denote the breadth of the gate, by b, and the weight, a unit of volume of the water, by w.

The pressure on EC will be equal to $\frac{1}{2}a^2bw$, and the pressure on EB will be equal to $\frac{1}{2}a'^2bw$; hence, the pressure on AC will be equal to

$$\tfrac{1}{2}bw\,(a^2 - a'^2)\,;$$

which is the pressure required.

From the principle of moments, the moment of the pressure on AC, is equal to the moment of the pressure on EC, minus the moment of the pressure on EB. Hence, from the last problem,

$$\tfrac{1}{2}bw\,(a^2 - a'^2) \times z = \tfrac{1}{2}bwa^2 \times \tfrac{2}{3}a - \tfrac{1}{2}bwa'^2 \times \tfrac{2}{3}a',$$

$$\therefore \quad z = \tfrac{2}{3}\,\frac{a^3 - a'^3}{a^2 - a'^2},$$

which is the required distance from the surface of the water.

This distance may be found directly from Equation (305).

3. Let it be required to find the pressure on a rectangular flood-gate, when both sides are pressed, the water being at different levels on the two sides. Also, to find the centre of pressure.

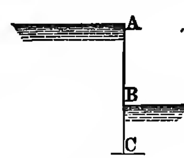

Fig. 137.

SOLUTION.—Denote the depth of water on one side by a, and on the other side by a', the other elements being the same as before.

The total pressure will, as before, be equal to,

$$\tfrac{1}{2}bw\,(a^2 - a'^2).$$

Estimating z from C upward, we have

$$z = \tfrac{1}{2}\,\frac{a^3 - a'^3}{a^2 - a'^2}. \quad \textit{Ans.}$$

4. A sluice-gate, 10 feet square, is placed vertically, its upper edge coinciding with the surface of the water. What is the pressure on the upper and lower halves of the gate, respectively, the weight of a cubic foot of water being taken equal to $62\frac{1}{2}$ *lbs.* ?

Ans. 7812.5 *lbs.*, and 23437.5 *lbs.*

5. What must be the thickness of a rectangular dam of granite, that it may neither rotate about its outer angular edge nor slide along its base, the weight of a cubic foot of granite being 160 *lbs.*, and the coefficient of friction between it and the soil being .6 ?

SOLUTION.—First, to find the thickness necessary to prevent rotation outward. Denote the height of the wall, by h, and suppose the water to extend from the bottom to the top. Denote the thickness, by t, and the length of the wall, or dam, by l. The weight of the wall in pounds, will be equal to

$$lht \times 160 ;$$

and this being exerted through its centre of gravity, the moment of the weight with respect to the outer edge, as an axis, will be equal to

$$\tfrac{1}{2}t^2lh \times 160 = 80lht^2.$$

The pressure of the water against the inner face, in pounds, is equal to

$$\tfrac{1}{2}lh^2 \times 62.5 = lh^2 \times 31.25.$$

This pressure is applied at the centre of pressure, which, Example 1, is at a distance from the bottom of the wall equal to $\frac{1}{3}h$; hence, its moment with respect to the outer edge of the wall, is equal to

$$lh^3 \times 10.4166.$$

The pressure of the water tends to produce rotation outward, and the weight of the wall acts to prevent this rotation. In order that these forces may be in equilibrium, their moments must be equal ; or

$$80lht^2 = lh^3 \times 10.4166.$$

Whence, we find, $\quad t = h\sqrt{.1302} = .36 \times h.$

Next, to find the thickness necessary to prevent sliding along the base. The entire force of friction due to the weight of the wall, is equal to

$$160lht \times .6 = 96lht ;$$

and in order that the wall may not slide, this must be equal to the pressure exerted horizontally against the wall. Hence,

$$96lht = 31.25lh^2.$$

Whence, we find, $$t = .325h.$$

If the wall is made thick enough to prevent rotation, it will be secure against sliding.

6. What must be the thickness of a rectangular dam 15 feet high, the weight of the material being 140 *lbs.* to the cubic foot, that, when the water rises to the top, the structure may be just on the point of overturning? *Ans.* 5.7 *ft.*

7. The staves of a cylindrical cistern filled with water are held together by a single hoop. Where must the hoop be situated?

Ans. At a distance from the bottom equal to one third of the height of the cistern.

8. Required the pressure of the sea on the cork of an empty bottle, when sunk to the depth of 600 feet, the diameter of the cork being $\frac{4}{5}$ of an inch, and a cubic foot of sea water being assumed to weigh 64 *lbs.*? *Ans.* 134 *lbs.*

9. Find the centre of pressure on a circular valve in a vertical floodgate, the radius of the valve being r, and the distance of its centre from the free surface being d. *Ans.* $y' = \frac{r^2}{4d} + d = \frac{r^2 + 4d^2}{4d}$.

10. A valve in a vertical gate is in the shape of an isosceles triangle whose base is parallel to the surface of the water, and whose altitude is 1 *ft.* How far from the free surface is the centre of pressure when the base is turned upward and when the distance from the surface to the base is 4 *ft.*? *Ans.* $y' = (\frac{1}{18} \div 4\frac{1}{3} + 4\frac{1}{3})$ *ft.* $= 4.346$ *ft.*

Buoyant Effort of Fluids.

170. Let A represent any solid body suspended in a heavy fluid. Conceive this solid to be divided into vertical prisms, whose horizontal sections are infinitely small. Any one of these prisms will be pressed downward by a force equal to the weight of a column of fluid, whose base (Art. 168) is equal to the hori-

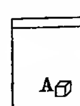

Fig. 138.

zontal section of the filament, and whose altitude is the distance of its upper surface from the surface of the fluid; it will be pressed upward by a force equal to the weight of a column of fluid having the same base and an altitude equal to the distance of the lower base of the filament from the surface of the fluid. The resultant of these two pressures is a force exerted vertically upward, and is equal to the weight of a column of fluid, equal in bulk to that of the filament and having its point of application at the centre of gravity of the volume of the filament. This being true for each filament of the body, and the lateral pressures being such as to destroy each other's effects, it follows that the resultant of all the pressures upon the body will be a vertical force exerted upward, whose intensity is equal to the weight of a portion of the fluid, whose volume is equal to that of the solid, and the point of application of which is the centre of gravity of the volume of the displaced fluid. This upward pressure is called **the buoyant effort** of the fluid, and its point of application is called **the centre of buoyancy.** The line of direction of the buoyant effort, in any position of the body, is called **a line of support.** That line of support which passes through the centre of gravity of a body is called **the line of rest.**

Floating Bodies.

171. A body wholly or partially immersed in a heavy fluid, is urged downward by its weight applied at its centre of gravity, and upward by the buoyant effort of the fluid applied at the centre of buoyancy.

The body will be in equilibrium when the line through the centre of gravity of the body, and the centre of buoyancy, is vertical; in other words, when the line of rest is vertical. When the weight of the body exceeds the buoyant effort, the body sinks to the bottom; when they are equal, it remains in

equilibrium, wherever placed. When the buoyant effort is greater than the weight, it rises to the surface, and, after a few oscillations, comes to rest, in such a position, that the weight of the displaced fluid is equal to that of the body, when it is said to **float.** The upper surface of the fluid is then called **the plane of flotation,** and its intersection with the surface of the body, **the line of flotation.**

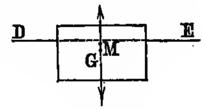

Fig. 139.

If a floating body be slightly disturbed from its position of equilibrium, the centres of gravity and buoyancy are no longer in the same vertical. Let DE represent the plane of flotation, G the centre of gravity of the body (Fig. 140), GH its line of rest, and C the centre of buoyancy.

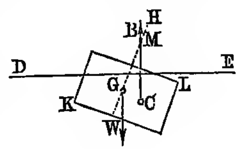

Fig. 140.

If the line of support, CB, intersect the line of rest in M, above G, as in Fig. 140, the buoyant effort and the weight conspire to restore the body to equilibrium; in this case the equilibrium is **stable.**

If M is below G, as in Fig. 141, the buoyant effort and the weight conspire to overturn the body; in this case the body, before being disturbed, must have been in **unstable** equilibrium.

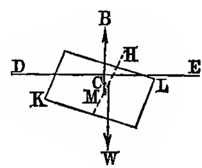

Fig. 141.

If the centres of buoyancy and gravity are always on the same vertical, M coincides with G (Fig. 142), and the body is in **indifferent** equilibrium. The limiting position of M, obtained by deflecting the body through an infinitely small angle, is the **meta-centre** of the body. Hence,

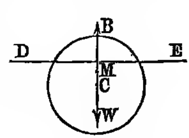

Fig. 142.

If the metacentre is above the centre of gravity, the body is in stable equilibrium; if below the centre of gravity, the body is in unstable equilibrium; if the points coincide, the body is in indifferent equilibrium.

The stability of a floating body is the greater, as the metacentre is higher above the centre of gravity. This condition is fulfilled in loading ships, by stowing the heavier objects near the bottom of the vessel.•

Specific Gravity.

172. The **specific gravity** of a body is its *relative weight;* that is, it is the number of times the body is heavier than an equivalent volume of some other body, taken as a standard.

The specific gravity of a body is obtained by dividing the weight of any volume of the body, by that of an equivalent volume of the standard.

For solids and liquids, distilled water is taken as a standard. Because this liquid is of different densities at different temperatures, it becomes necessary to assume a standard temperature for it; for a like reason, a standard temperature must be taken for the body whose specific gravity is to be found. Different standards of temperature have been assumed by different writers; we shall adopt those assumed by Jamin, who takes for the standard temperature of water, 4° C., or about 39° F., and for the standard temperature of the body, 0° C., or 32° F. The former is the temperature at which water has a maximum density, and the latter is that of melting ice.

In finding the specific gravity of a body, we first determine it with respect to water at any temperature; this we may call the **observed specific gravity.** We then correct the result for the temperature of the water, by means of a table of densities of water at different temperatures, that at 39° being 1; this result we call the **apparent specific gravity**.

Finally, we correct this for the temperature of the body, and thus find the **true specific gravity.**

1st. Let d be the density of water at the temperature t, its density at 39° being 1; let s be the *observed* specific gravity of a body referred to water at the temperature t, and let s' be its specific gravity referred to water at 39° F.

Because the specific gravity of a body varies inversely as the density of the water to which it is referred, we have,

$$s : s' :: 1 : d; \quad \therefore \quad s' = ds \quad . \quad . \quad . \quad . \quad . \quad (308)$$

That is, *to find the apparent specific gravity of a body, multiply its observed specific gravity, at the temperature t, by the corresponding tabular density of water.*

2dly. Suppose the body to have the same temperature, t, as the water to which it is referred. Denote the volume of the body at the temperature t, by v', and at 32° by v; denote the corresponding specific gravities by s' and s.

Because the specific gravity of a body of given weight varies inversely as its volume, we have,

$$s : s' :: v' : v; \quad \therefore \quad s = s'\frac{v'}{v} \quad . \quad . \quad . \quad . \quad . \quad (309)$$

That is, *to find the true specific gravity of a body, multiply its apparent specific gravity by the quotient of its volume at the temperature t, by its volume at* 32°.

This quotient may be found from the body's known rate of expansion.

It is only in nice determinations that it is necessary to take account of the latter correction.

Gases are usually referred to air as a standard; but as air is easily referred to water, we may take distilled water at 39° F. as a standard for all bodies.

Sometimes it is convenient to find the specific gravity of a

body with respect to some other body whose specific gravity is already known. In this case the required specific gravity is equal to the product of that which is found, by that which is already known. Thus, if A is m times as heavy as B, and if B is n times as heavy as C, then will A be mn times as heavy as C.

Methods of Finding Specific Gravity.

173. There are two principal methods of finding the specific gravity of a body; *first*, by means of the balance, and *secondly*, by means of the hydrometer. The former alone can be used for nice determinations, such as are needed in the operations of analytical chemistry and physics; the latter is of easier application, and is sufficiently accurate for most practical purposes.

Hydrostatic Balance.

174. This balance is similar to that described in Article 68; the scale-pans, however, are provided with hooks for suspending bodies, as shown in the figure.

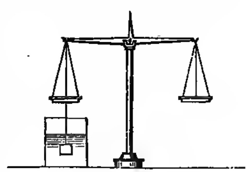

Fig. 143.

In balances of modern construction the vessel containing water is placed on a movable bench or shelf, that strides one of the scale-pans, without interfering with its movements, and the body is then suspended from the beam by a thread or wire. In both cases a body attached to the string may be weighed either in the air or in the water, at pleasure.

Specific Gravity of an Insoluble Body.

175. Fasten the suspending wire to one scale-pan, or to one extremity of the beam, as the case may be, and counterpoise it by weights in the opposite pan. Then attach the body to the wire and counterpoise it by weights in the other pan; *these give the weight of the body in air*; next immerse

the body in water, so as not to touch the containing vessel; the buoyant effort of the water will thrust the body up with a force equal to the weight of the displaced water, and restore the equilibrium by weights placed in the first pan; *these will give the weight of the displaced water:* divide the weight of the body in air by the weight of the displaced water, and the quotient will be the observed specific gravity.

Thus, if a piece of copper weigh 2047 grains in air, and lose 230 grains when weighed in water, its specific gravity is $\frac{2047}{230}$, or 8.9.

If the body will not sink in water, determine its weight in air, as before; then attach to it a body so heavy that the combination will sink; find the weight of the water displaced by the combination, and also the weight displaced by the heavy body, take their difference, and the result will be the weight of the water displaced by the body in question; then proceed as before.

Thus, a body weighs 600 grains in air; when attached to a piece of copper, the combination weighs 2647 grains in air, and suffers a loss of 834 grains in water, the copper alone losing 230 grains. The buoyant effort of the fluid exerted on the body is therefore 604 grains, and the specific gravity of the body is $\frac{600}{604}$, or 0.993.

Specific Gravity of a Soluble Body.

176. Find its specific gravity with respect to some liquid in which it is not soluble; find also the specific gravity of this liquid with respect to water; take the product of these, and it will be the specific gravity sought (Art. 172).

Thus, if the specific gravity of a body with respect to oil be 3.7, and the specific gravity of the oil with respect to water be 0.9, the specific gravity of the body is 3.7×0.9, or 3.33.

It is often convenient to use a saturated solution of the substance in question as the auxiliary liquid.

Specific Gravity of Liquids.

177. 1*st Method.*—The most convenient method is by the specific gravity bottle. This is a bottle constructed to hold exactly 1000 grains of distilled water. Accompanying it is a brass weight, just equal to the empty bottle. To use it, let it be filled with the liquid in question, and placed in one scale-pan; in the other pan place the brass counterpoise, and weights enough to balance the liquid; divide the number of grains in the weight of the liquid by 1000, and the quotient will be the specific gravity.

Thus, if the bottle filled with a liquid weighs 945 grains, beside the counterpoise, its specific gravity is 0.945.

2*d Method.*—Take a body, that will sink both in the liquid and in water, and which is not acted upon by either; determine its loss of weight, first in the liquid, then in water; divide the former by the latter, and the quotient will be the specific gravity sought. The reason is evident.

Thus, if a glass ball lose 30 grains when weighed in water, and 24 in alcohol, the specific gravity of the alcohol is $\frac{24}{30}$, or 0.8.

Hydrometers.

178. An **hydrometer** is a floating body, used in finding specific gravities. Its construction depends on the principle of flotation. Hydrometers are of two kinds. 1°. Those in which the submerged volume is constant. 2°. Those in which the weight of the instrument is constant.

Nicholson's Hydrometer.

179. This instrument consists of a hollow cylinder, *A*, at the lower extremity of which is a basket, *B*, and at the upper extremity a wire, bearing a scale-pan, *C*. At the bottom of the basket is a ball, *E*, containing mercury, to cause the instrument to float in an upright position. By means of this

ballast, the instrument is adjusted so that a given weight, say 500 grains, placed in the pan, *C*, will sink it in distilled water to a notch, *D*, filed in the neck.

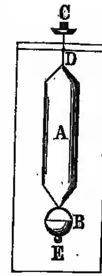

Fig. 144.

This instrument is in reality a **weighing-machine**, and as such can be used for determining the approximate weights of bodies within certain limits; in the instrument described, no body can be weighed whose weight exceeds 500 grains.

To find the specific gravity of a solid, place it in the pan, *C*, and add weights till the instrument sinks, in distilled water, to the notch, *D*. The added weights, subtracted from 500 grains, give the weight of the body in air. Place the body in the basket, *B*, which generally has a reticulated cover, to prevent the body from floating away, and add other weights to the pan, until the instrument again sinks to the notch, *D*. The weights last added give the weight of water displaced by the body. Divide the first of these by the second, and the quotient will be the specific gravity required.

To find the specific gravity of a liquid. Having weighed the instrument, place it in the liquid, and add weights to the scale-pan, till it sinks to *D*. The weight of the instrument, plus the weight added, will be the weight of the liquid displaced by the instrument. The weight of the instrument added to 500 grains gives the weight of an equal volume of distilled water. The quotient of the first by the second is the specific gravity required.

A modification of this instrument, in which the basket, *B*, is omitted, is sometimes used for determining specific gravities of liquids only. This kind of hydrometer, known as Fahrenheit's hydrometer, is generally made of glass, that it may not be acted on chemically by the liquids into which it is plunged.

Scale Areometer.

180. The **scale areometer** is a hydrometer whose weight is constant; the specific gravity of a liquid is made known by the depth to which it sinks in it. The instrument consists of a glass cylinder, *A*, with a stem, *C*, of uniform diameter. At the bottom of the cylinder is a bulb, *B*, containing mercury, to make the instrument float upright. By introducing a suitable quantity of mercury, the instrument may be adjusted so as to float at any desired point of the stem.

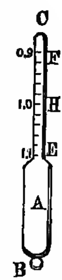

Fig. 145.

When it is designed to determine the specific gravity of liquids, both lighter and heavier than distilled water, it is called a **universal hydrometer**, and is so ballasted as to float in distilled water at the middle of the stem. This point is marked on the stem, and is numbered 1 on the scale. A liquid is then formed, by dissolving salt in water, whose specific gravity is 1.1, and the instrument is allowed to float freely in it; the point, *E*, to which it sinks, is marked on the stem, and the intermediate part of the scale, *HE*, is divided into 10 equal parts. In like manner a mixture of alcohol and water is formed, whose specific gravity is 0.9, the corresponding position of the plane of flotation is marked on the stem, and the space between it and the division 1 is divided into 10 equal parts. The graduation is continued, both up and down, through the whole length of the stem. The graduation is marked on a piece of paper within the stem.

To use this hydrometer, we put it into the liquid and allow it to come to rest; the division of the scale that corresponds to the surface of flotation shows the specific gravity of the liquid. The hypothesis on which this instrument is graduated, is, that the increments of specific gravity are proportional to the increments of the submerged portion of the

stem. This hypothesis is only approximately true, but it approaches more nearly to the truth as the diameter of the stem diminishes.

The Alcoometer.

181. This instrument is similar in construction to the scale areometer; the graduation, however, is made on a different principle. Its object is to determine the percentage of alcohol in a mixture of alcohol and water. The graduation is made as follows: the instrument is first placed in absolute alcohol, and ballasted so that it will sink nearly to the top of the stem. This point is marked 100. Next, a mixture of 95 parts of alcohol and 5 of water is made, and the point to which the instrument sinks is marked 95. The intermediate space is divided into 5 equal parts. Next, a mixture of 90 parts of alcohol and 10 of water is made; the point to which the instrument sinks is marked 90, and the space between this and 95 is divided into 5 equal parts. In this manner, the entire stem is graduated by successive operations. The spaces on the scale are not equal at different points, but, for a space of five parts, they may be so regarded, without sensible error.

To use the instrument, place it in the mixture of alcohol and water, and read the division to which it sinks; this will indicate the percentage of alcohol in the mixture.

In all these instruments, the temperature has to be taken into account; this is effected by tables that accompany the different instruments.

On the principle of the alcoometer, a great variety of areometers are constructed, for determining the strength of wines, syrups, and other liquids employed in the arts.

In some nicely constructed hydrometers, the mercury used as ballast serves also to fill the bulb of a delicate thermometer, whose stem rises into the cylinder of the instrument, and

thus enables us to note the temperature of the fluid in which it is immersed.

EXAMPLES.

1. A cubic foot of water weighs 1000 ounces. Required the weight of a cubical block of stone, whose edge is 4 feet, its specific gravity being 2.5. *Ans.* 10,000 *lbs.*

2. Required the number of cubic feet in a body whose weight is 1000 *lbs.*, its specific gravity being 1.25. *Ans.* 12.8.

3. Two lumps of metal weigh 3 *lbs.*, and 1 *lb.*, and their specific gravities are 5 and 9. What will be the specific gravity of an alloy formed by melting them together, supposing no contraction of volume to take place? *Ans.* 5.625.

4. A body weighing 20 grains has a specific gravity of 2.5. Required its loss of weight in water. *Ans.* 8 grains.

5. A body weighs 25 grains in water, and 40 grains in a liquid whose specific gravity is .7. What is the weight of the body in a vacuum? *Ans.* 75 grains.

6. A NICHOLSON'S hydrometer weighs 250 grains, and it requires an additional weight of 336 grains to sink it to the notch in the stem, in a mixture of alcohol and water. What is the specific gravity of the mixture? *Ans.* .781.

7. A block of wood sinks in distilled water till $\frac{7}{8}$ of its volume is submerged. What is its specific gravity? *Ans.* .875.

8. The weight of a piece of cork in air, is $\frac{3}{4}$ *oz.*; the weight of a piece of lead in water, is $6\frac{4}{9}$ *oz.*; the weight of the cork and lead together in water, is $4\frac{7}{100}$ *oz.* What is the specific gravity of the cork? *Ans.* 0.24.

9. A solid, whose weight is 250 grains, weighs in water, 147 grains, and, in another fluid, 120 grains. What is the specific gravity of the latter fluid? *Ans.* 1.262.

10. A solid weighs 60 grains in air, 40 in water, and 30 in an acid. What is the specific gravity of the acid? *Ans.* 1.5.

11. A wooden sphere whose diameter is 8 inches, and whose specific gravity is .75, is placed in a vessel of distilled water. To what depth will it sink? *Ans.* 5.55 inches, nearly.

The following table is compiled from the *Ordnance Manual:*

TABLE OF SPECIFIC GRAVITIES OF SOLIDS AND LIQUIDS.

OLIDS.	SPEC. GRAV.	SOLIDS.	SPEC. GRAV.
Antimony, cast	6.712	Limestone	3.180
Brass, cast	8.396	Marble, common	2.686
Copper, cast	8.788	Salt, common	2.130
Gold, hammered	19.361	Sand	1.800
Iron, bar	7.788	Slate	2.612
Iron, cast	7.207	Stone, common	2.520
Lead, cast	11.352	Tallow	0.945
Mercury at 32° F	13.598	Boxwood	0.912
" at 60°	13.580	Cedar	0.596
Platina, rolled	22.069	Cherry	0.715
" cast	20.337	Lignum vitæ	1.333
Silver, hammered	10.511	Mahogany	0.854
Tin, cast	7.291	Oak, heart	1.170
Zinc, cast	6.861	Pine, yellow	0.660
Bricks	1.900	Nitric acid	1.217
Chalk	2.784	Sulphuric acid	1.841
Coal, bituminous	1.270	Alcohol, absolute	0.792
Diamond	3.521	Ether, sulphuric	0.715
Earth, common	1.500	Sea water	1.026
Gypsum	2.168	Olive oil	0.915
Ivory	1.822	Oil of Turpentine	0.870

Velocity of a Liquid Jet Through a Small Orifice.

182. When the motion of a liquid through a channel is such that the particles passing any point always move in the same direction and with the same velocity, the flow is said to be **steady.** In treating of **steady flow** it is assumed that the liquid conforms to **the law of continuity**, that is, that its particles remain in contact, without hollow, or interval. If the channel has a varying cross section, the average velocity must vary from section to section, and in order that the quantity that flows through the different sections in any given time may be constant, the average velocities at different points of the channel must vary inversely as the cross-sections at those points.

Let the figure represent a vessel having a small orifice, D, at the bottom, and suppose it to be kept filled with some liquid up to the level AB, the height of AB above D being denoted by h. When the flow has become *steady*, denote the velocity at the level of AB by v_1, and the velocity at the orifice by v; also, denote the area of the cross-section, AB, by A, and that of the orifice by a. We then have, from what precedes,

Fig 146.

$$v_1 = \frac{a}{A} v \quad . \quad . \quad . \quad . \quad . \quad (310)$$

Let us denote the mass of the fluid that flows out at the orifice in one second by m. As this fluid escapes, there will be a subsidence of the entire fluid to preserve the law of continuity, and it is obvious that the work performed by the weight of this descending mass is equal to that which would be performed by the weight of the mass, m, in descending from the level, AB, to D; that is, to mgh.

The kinetic energy of the mass, m, when it flows out at D, is equal to $\frac{1}{2}mv^2$, and of the same mass at the level, AB, is $\frac{1}{2}mv_1^2$; hence, the gain of kinetic energy due to the work, mgh, is equal to $\frac{1}{2}m(v^2 - v_1^2)$. Equating these values, we have,

$$mgh = \tfrac{1}{2}m(v^2 - v_1^2), \quad \text{or,} \quad 2gh = v^2 - v_1^2 \quad . \quad . \quad . \quad . \quad . \quad (311)$$

Substituting the value of v_1, from (310), and reducing, we have,

$$v = \sqrt{\frac{2gh}{1 - \frac{a^2}{A^2}}} \quad . \quad . \quad . \quad . \quad . \quad (312)$$

If the orifice is extremely small as compared with the section AB, a^2 will be still smaller in comparison with A^2, and we have, approximately,

$$v = \sqrt{2gh} \quad . \quad . \quad . \quad . \quad . \quad (313)$$

Hence, *a liquid issues from a very small orifice in the bottom of a vessel, with a velocity equal to that acquired by a body in falling through a height equal to the distance of the orifice below the free surface.*

We have seen that the pressure due to the weight of a fluid on any point of the surface of a vessel, is normal to the surface, and is proportional to the depth of the point below the free surface. Hence, if an orifice be made at any point, the liquid will flow out in a jet, normal to the surface at that point, and with a velocity due to the depth of the orifice below the free surface of the fluid.

If the orifice is on a vertical side of a vessel, the initial direction of the jet will be horizontal; if it be at a point where the tangent plane is oblique to the horizon, the initial direction of the jet will be oblique; if the opening is on the upper side of a portion of a vessel where the tangent is horizontal, the jet will be directed upward, and will rise to a height due to the velocity; that is, to the height of the upper surface of the fluid.

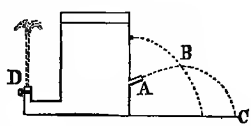

Fig. 147.

The value of v, in (313), is only *approximate.* The law of steady flow implies that there is no abrupt change in the cross section of the channel; in the case supposed there is such a change at the outlet, and for this reason the theoretical value of v requires some modification, as will be seen hereafter.

Modification Due to Extraneous Pressure.

183. If the upper surface of the liquid, in any of the preceding cases, be pressed by a force, as when it is urged downward by a piston, we may denote the height of a column of the fluid whose weight is equal to the extraneous pressure, by

h'. The velocity of efflux will then be given by the equation,

$$v = \sqrt{2g(h + h')}. \quad . \quad . \quad . \quad . \quad . \quad (314)$$

The pressure of the atmosphere acts equally on the upper surface and the opening; hence, in ordinary cases, it may be neglected; but were the liquid to flow into a vacuum, or into rarefied air, the pressure must be taken into account, and this may be done by means of the formula just given.

Should the flow take place into condensed air, or into any medium which opposes a greater resistance than the atmospheric pressure, the extraneous pressure would act upward, h' would be negative, and the preceding formula would become,

$$v = \sqrt{2g(h - h')}. \quad . \quad . \quad . \quad . \quad . \quad (315)$$

Spouting of Liquids on a Horizontal Plane.

184. Let KL be a vessel filled with water, D an orifice in its vertical side, and DE the path of the spouting fluid. We may regard each drop as a projectile shot forth horizontally, and then acted on by gravity. Its path is, therefore, a parabola, and the circumstances of its motion are made known by equations (169) and (170). In Eq. (170) h represents LD, h' represents DK, and x represents KE.

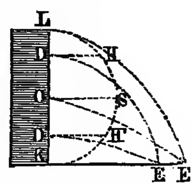

Fig. 148.

If we describe a semicircle on KL, and through D draw an ordinate, DH, we have, from a property of the circle,

$$DH = \sqrt{DK \cdot DL} = \sqrt{hh'}.$$

Hence we have, by substitution,

$$KE = 2DH.$$

Since there are two points on KL at which the ordinates are equal, there must be two orifices through which the fluid will spout to the same distance on the horizontal plane KE; one of these is as far above the centre, O, as the other is below it.

If the orifice be at O, midway between K and L, the ordinate, OS, will be greatest possible, and the range, KE', will be a maximum. The range in this case will be equal to the diameter of the circle, LHK, or to the distance from the surface of the water in the vessel to the horizontal plane.

Let a semi-parabola, LE', be described, having its axis, LK, vertical, its focus at K, and its vertex at L; then if we suppose a jet to issue from K, being directed upward at different angles by a short pipe, it may be shown, as in Art. 108, that every point within the curve may be reached by two jets, every point on the curve may be reached by one jet, and that points lying without the curve cannot be reached at all.

A jet thrown obliquely upward, as shown at A in Fig. 147, is an arc of a parabola, since each particle may be regarded as a projectile; the circumstances of motion of the particles will be made known by Equation (160). In like manner, the same equation will make known the circumstances of motion when a jet is projected obliquely downward.

Coefficients of Efflux and Velocity.

185. When a vessel empties itself through a small orifice at its bottom, it is observed that the particles of fluid near the top descend in vertical lines; when they approach the bottom they incline toward the orifice, the converging lines of fluid particles tending to cross each other as they emerge from the vessel. The result is, that the stream grows narrower, after leaving the vessel, until it reaches a point at a distance from the vessel equal to about the radius of the

orifice, when the contraction becomes a minimum, and below that point the vein again spreads out. This phenomenon is called **the contraction of the vein.** The cross-section at the most contracted part of the vein, is not far from $\frac{64}{100}$ of the area of the orifice, when the vessel is very thin. If we denote the area of the orifice, by a, and the area of the least cross-section of the vein, by a', we shall have,

$$a' = ka,$$

in which k is a number to be determined by experiment. This number is called the **coefficient of contraction.**

To find the quantity of water discharged through an orifice at the bottom of the containing vessel, in one second, we have only to multiply the area of the smallest cross-section of the vein, by the velocity. Denoting the quantity discharged in one second, by Q', we shall have,

$$Q' = ka\sqrt{2gh}.$$

This formula is only true on the supposition that the actual velocity is equal to the theoretical velocity, which is not the case, as has been shown by experiment. The theoretical velocity has been shown to be equal to $\sqrt{2gh}$, and if we denote the actual velocity, by v', we shall have,

$$v' = l\sqrt{2gh},$$

in which l is to be determined by experiment; this value of l is slightly less than 1, and is called the **coefficient of velocity.** In order to get the actual discharge, we must replace $\sqrt{2gh}$ by $l\sqrt{2gh}$, in the preceding equation. Doing so, and denoting the actual discharge per second, by Q, we have,

$$Q = kla\sqrt{2gh}.$$

The product kl, is called the **coefficient of efflux.** It

has been shown by experiment, that this coefficient for orifices in thin plates, is not quite constant. It decreases slightly, as the area of the orifice and the velocity are increased; and it is further found to be greater for circular orifices than for those of any other shape.

If we denote the coefficient of efflux, by m, we have,

$$Q = ma\sqrt{2gh} \quad . \quad . \quad . \quad . \quad . \quad (316)$$

In this equation, h is called **the head of water.** Hence, we may define the *head of water* to be the distance from the orifice to the plane of the upper surface of the fluid.

The mean value of m corresponding to orifices of from $\frac{1}{2}$ an inch to 6 inches in diameter, with from 4 to 20 feet head of water, has been found to be about .615. If we take the value of $k = .64$, we shall have,

$$l = \frac{m}{k} = \frac{.615}{.640} = .96.$$

That is, the actual velocity is only $\frac{96}{100}$ of the theoretical velocity. This diminution is due to friction, viscosity, etc.

Efflux Through Short Tubes.

186. It is found that the discharge from a given orifice is increased, when the thickness of the plate through which the flow takes place, is increased; also, when a short tube is introduced.

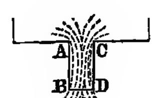

Fig. 149.

When a tube AB, is employed which is not more than four times as long as the diameter of the orifice, the value of m becomes, on an average, equal to .813; that is, the discharge per second is 1.325 times greater when the tube is used, than without it. In using the cylindrical tube, the contraction takes place at the outlet of the vessel, and not at the outlet of the tube.

Compound mouth-pieces are sometimes used formed of two conic frustums, as shown in the figure, having the form of the vein. It has been shown by ETELWEIN, that the most effective tubes of this form should have the diameter of the cross-section CD, equal to .833 of the diameter AB. The angle made by the sides CF and DE, should be about 5° 9′, and the length of this portion should be three times that of the other.

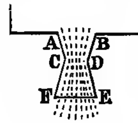

Fig. 150.

EXAMPLES.

1. With what theoretical velocity will water issue from a small orifice $16\frac{1}{12}$ feet below the surface of the fluid ? *Ans.* $32\frac{1}{6}$ *ft.*

2. If the area of the orifice, in the last example, is $\frac{1}{10}$ of a square foot, and the coefficient of efflux .615, how many cubic feet of water will be discharged per minute? *Ans.* 118.695 *cu. ft.*

3. A vessel, constantly filled with water, is 4 feet high, with a cross-section of one square foot; an orifice in the bottom has an area of one square inch. In what time will three fourths of the water be drawn off, the coefficient of efflux being .6 ? *Ans.* $\frac{3}{4}$ of a minute, nearly.

4. A vessel is kept constantly full of water. How many cubic feet of water will be discharged per minute from an orifice 9 feet below the upper surface, having an area of 1 square inch, the coefficient of efflux being .6 ? *Ans.* 6 cubic feet, about.

5. In the last example, what will be the discharge per minute, if we suppose each square foot of the upper surface to be pressed by a force of 645 *lbs.*? *Ans.* $8\frac{3}{4}$ cubic feet, about.

6. The head of water is 16 feet, and the orifice is $\frac{1}{100}$ of a square foot. What quantity of water will be discharged per second, when the orifice is through a thin plate?

SOLUTION.—In this case, we have,

$$Q = .615 \times .01 \sqrt{2 \times 32\tfrac{1}{6} \times 16} = .197 \text{ cubic feet.}$$

When a short cylindrical tube is used, we have,

$$Q = .197 \times 1.325 = .261 \text{ cubic feet.}$$

In ETELWEIN's compound mouth-piece, if we take the smallest cross-

section as the orifice, and denote it by a, it is found that the discharge is $2\frac{1}{2}$ times that through an orifice of the same size in a thin plate. In this case, we have, supposing $a = \frac{1}{100}$ of a square foot,

$$Q = .197 \times 2\tfrac{1}{2} = .49 \text{ cubic feet.}$$

Time Required for a Vessel to Empty Itself.

187. Let it be required to find the time in which a vessel will empty itself through a small orifice in its bottom. Assume a horizontal plane of reference through the orifice, and denote the distance from this plane to the free surface by z; denote the area of the free surface by S, and suppose this surface to be depressed by the outflow through a distance dz in the time dt; the entire volume of the outflow in the time dt will then be equal to Sdz. But the effective velocity of outflow is equal to $m\sqrt{2gz}$; if we denote the area of the orifice by a, we shall therefore have for the volume of the outflow $ma\sqrt{2gz}\,dt$. Equating these values, we have

$$ma\sqrt{2gz}\,dt = Sdz, \text{ or } dt = -\frac{S}{ma\sqrt{2g}} \cdot \frac{dz}{z^{\frac{1}{2}}}, \quad . \quad . \quad . \quad . \quad . \quad (317)$$

in which the second member is made negative because z diminishes as t increases.

Integrating (317) from h the original height of the liquid above the orifice to 0, we have

$$t = -\frac{1}{ma\sqrt{2g}}\int_h^0 \frac{Sdz}{z^{\frac{1}{2}}}, \quad . \quad . \quad . \quad . \quad . \quad (318)$$

which gives the required time.

Examples.

1. Let S be constant and equal to A: in this case we have from (318), for the time required for the vessel to empty itself,

$$t = \frac{2A\sqrt{h}}{ma\sqrt{2g}} = \frac{A}{ma}\sqrt{\frac{2h}{g}}, \quad . \; . \; . \; . \; . \; (319)$$

in which m may be taken equal to .615.

2. Let the surface be a surface of revolution whose axis is vertical, and let the equation of its meridian curve be $x^4 = p^4 z$. In this case we have $S = \pi x^2 = \pi p^2 z^{\frac{1}{2}}$; substituting in (318), we have

$$t = -\frac{1}{ma\sqrt{2g}}\int_h^0 \pi p^2 dz \quad . \; . \; . \; . \; . \; (320)$$

Performing the integration from h to z, we have

$$t = \frac{\pi p^2}{ma\sqrt{2g}}(h - z) \quad . \; . \; . \; . \; . \; (321)$$

From which we see that t varies as the height through which the free surface falls.

Making $z = 0$, in (320), we have for the time required for the vessel to empty itself,

$$t = \frac{\pi p^2 h}{ma\sqrt{2g}} \quad . \; . \; . \; . \; . \; (322)$$

Other examples may be solved in a similar manner.

Motion of Water in Open Channels.

188. When water flows through an open channel, as in a river, canal, or open aqueduct, the form of the channel being always the same, and the supply of water being constant, it is a matter of observation that the flow becomes steady ; that is, the quantity of water that flows through any cross-section, in a given time, is constant. On account of adhesion, friction, etc., the particles of water next the sides and bottom of the channel have their motion retarded. This retardation is imparted to the next layer of particles, but in a less degree, and so on, till a line of particles is reached

whose velocity is greater than that of any other filament. This line, or filament of particles, is called the **axis of the stream.** In the case of cylindrical pipes, the axis coincides sensibly with the axis of the pipe; in straight, open channels, it coincides with that line of the upper surface which is midway between the sides.

A section at right angles to the axis is called a **cross-section**, and, from what has been shown, the velocities of the fluid particles will be different at different points of the same cross-section. The **mean velocity** corresponding to any cross-section, is the average velocity of the particles at every point of that section. The mean velocity may be found by dividing the volume which flows through the section in one second, by the area of the cross-section. Since the same volume flows through each cross-section per second, after the flow has become uniform, it follows that, in channels of varying width, the mean velocity, at any section, will be inversely as the area of the section.

The intersection of the plane of cross-section with the sides and bottom of the channel, is called the **perimeter** of the section. In the case of a pipe which is constantly filled, the perimeter is the entire line of intersection of the plane of cross-section, with the interior surface of the pipe.

The mean velocity of water in an open channel depends, in the first place, upon its inclination to the horizon. As the inclination becomes greater, the component of gravity in the direction of the channel increases, and, consequently, the velocity becomes greater. Denoting the inclination by I, and resolving the force of gravity into two components, one at right angles to the upper surface, and the other parallel to it, we shall have for the latter component,

$$g \sin I.$$

This is the only force that acts to increase the velocity.

The velocity will be diminished by friction, adhesion, etc. The total effect of these resistances will depend upon the ratio of the perimeter to the area of the cross-section, and also upon the velocity. The cross-section being the same, the resistances will increase as the perimeter increases; consequently, for the same cross-section, the resistance of friction will be the least possible when the perimeter is the least possible. The retardation of the flow will also diminish as the area of the cross-section is increased, other things remaining unchanged.

If we denote the area of the cross-section by a, the perimeter by P, and the velocity by v, we shall have,

$$\frac{ag \sin I}{P} = f(v),$$

in which $f(v)$ denotes some function of v.

Since the inclination is very small in all practical cases, we may place the inclination itself for the sine of the inclination, and doing so, it has been shown by PRONY, that the function of v may be expressed by two terms, one of which is of the first, and the other of the second degree, with respect to v; or,

$$\frac{ga\,I}{P} = mv + nv^2.$$

Denoting $\frac{a}{P}$ by R, $\frac{m}{g}$ by k, and $\frac{n}{g}$ by l, we have, finally,

$$kv + lv^2 = RI,$$

in which k and l are constants, to be determined by experiment. According to ETELWEIN, we have,

$$k = .0000242651, \quad \text{and} \quad l = .0001114155.$$

Substituting these values, and solving with respect to v, we have,

$$v = -0.1088941604 + \sqrt{.0118580490 + 8975.414285RI},$$

from which the velocity can be found when R and I are known. The values of k and l, and consequently that of v, were found by PRONY to be somewhat different from those given above. Those of ETELWEIN are selected for the reason that they were based upon a much larger number of experiments than those of PRONY.

Having the mean velocity and the area of the cross-section, the quantity of water delivered in any time can be computed. Denoting the quantity delivered in n seconds by Q, and retaining the preceding notation, we have,

$$Q = nav. \quad . \quad . \quad . \quad . \quad . \quad (323)$$

The quantity of water to be delivered is generally one of the data in all practical problems involving the distribution of water. The difference of level of the point of supply and delivery is also known. The preceding principles enable us to give such a form to the cross-section of the canal, or aqueduct, as will insure the requisite supply.

Were it required to apply the results just deduced, to the case of irregular channels, or to those in which there were many curves, a considerable modification would be required. The theory of these modifications does not come within the limits assigned to this treatise. For a complete discussion of the whole subject of hydraulics in a popular form, the reader is referred to the *Traité d'Hydraulique* D'AUBISSON.

Motion of Water in Pipes.

189. The circumstances of the motion of water in pipes, are closely analogous to those of its motion in open channels.

The forces which tend to impart motion are dependent upon the weight of the water in the pipe, and upon the height of the water in the upper reservoir. Those which tend to prevent motion depend upon the depth of water in the lower reservoir, friction in the pipe, adhesion, and shocks arising from irregularities in the bore of the pipe. The retardation due to shocks will, for the present, be neglected.

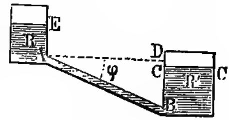

Fig. 151.

Let AB represent a straight cylindrical pipe, connecting two reservoirs R and R'. Suppose the water to maintain its level at E, in the upper, and at C, in the lower reservoir. Denote AE by h, and BC by h'. Denote the length of the pipe by l, its circumference by c, its cross-section by a, its inclination by ϕ, and the weight of a unit of volume of water by w.

Experience shows that, under the circumstances above indicated, the flow soon becomes uniform. We may then regard the entire mass of fluid in the pipe as a coherent solid, moving with a mean uniform velocity down the inclined plane AB.

The weight of the water in the pipe will be equal to wal. If we resolve this weight into two components, one perpendicular to, and the other coinciding with the axis of the tube, we shall have for the latter component, $wal \sin \phi$. But $l \sin \phi$ is equal to DB. Denoting this distance by h'', we shall have for the pressure in the direction of the axis, due to the weight of the water in the pipe, the expression wah''. This pressure acts from A toward B. The pressure due to the weight of the water in R, and acting in the same direction, is wah.

The forces acting from B toward A, are, *first*, that due to the weight of the water in R', which is equal to wah'; and, *secondly*, the resistance due to friction and adhesion. This resistance depends upon the length of the pipe, its circumfer-

ence and the velocity. It has been shown, by experiment, that this force may be expressed by the term,

$$cl\,(kv + k'v^2).$$

Since the velocity has been supposed uniform, the forces acting in the direction of the axis must be in equilibrium. Hence,

$$wah + wah'' = wah' + cl\,(kv + k'v^2)\,;$$

whence, by reduction,

$$\frac{k}{w}v + \frac{k'}{w}v^2 = \frac{a}{c}\left(\frac{h + h'' - h'}{l}\right).$$

The factor $\frac{a}{c}$ is equal to one fourth of the diameter of the pipe. Denoting this by d, we shall have, $\frac{a}{c} = \frac{1}{4}d$; denoting $\frac{k}{w}$ by m, $\frac{k'}{w}$ by n, and $\frac{h + h'' - h'}{l}$ by s, we have,

$$mv + nv^2 = \tfrac{1}{4}sd. \quad . \quad . \quad . \quad . \quad . \quad (324)$$

The values of m and n, as determined experimentally by PRONY, are,

$$m = 0.00017, \qquad \text{and} \qquad n = 0.000106.$$

Hence, by substitution,

$$.00017v + .000106v^2 = \tfrac{1}{4}sd.$$

If v is not very small, the first term may be neglected, which will give,

$$v = 48.56\sqrt{sd}.$$

If we denote the quantity of water delivered in n seconds by Q, we shall have,

$$Q = nav = 48.56\,na\sqrt{sd}. \quad . \quad . \quad . \quad . \quad . \quad (325)$$

The velocity will be greatly diminished if the tube is curved to any considerable extent, or if its diameter is not uniform throughout. It is not intended to enter into a discussion of these cases; their complete development would require more space than has been allotted to this branch of Mechanics.

General Remarks on the Distribution and Flow of Water in Pipes.

190. Whenever an obstacle occurs in the course of an open channel or pipe, a change of velocity must take place. In passing the obstacle, the velocity of the water will increase, and then, impinging upon that which has already passed, a shock will take place. This shock consumes a certain amount of kinetic energy, and thus diminishes the velocity of the stream. All obstacles should be avoided; or, if any are unavoidable, the stream should be diminished, and again enlarged gradually, so as to avoid, as much as possible, the necessary shock incident to sudden changes of velocity.

For a like reason, when a branch enters the main channel, it should be made to enter as nearly in the direction of the current as possible.

All changes of direction give rise to mutual impacts amongst the particles, and the more, as the change is more abrupt. Hence, when a change of direction is necessary, the straight branches should be made tangential to the curved portion.

The entrance to, and outlet from a pipe or channel, should be enlarged, in order to diminish, as much as possible, the coefficients of ingress and egress.

When a pipe passes over uneven ground, sometimes ascending, and sometimes descending, there is a tendency to a collection of bubbles of air, at the highest points, which may finally come to act as an impeding cause to the flow. There

should, therefore, be suitable pipes inserted at the highest points, to permit the confined air to escape.

Finally, attention should be given to the form of the cross-section of the channel. If the channel is a pipe, it should be made cylindrical. If it is a canal or open aqueduct, that form should be given to the perimeter which would give the greatest cross-section, and, at the same time, conform to the necessary conditions of the structure. The perimeter in open channels is generally trapezoidal, from the necessity of the case; and it should be remembered, that the nearer the form approaches a semicircle, the greater will be the flow.

IX.—MECHANICS OF GASES AND VAPORS.

Gases and Vapors.

191. Gases and **vapors** are distinguished from other fluids by their great compressibility and correspondingly great expansibility. They continually tend to expand, and if left free the expansion will go on till counteracted by some extraneous force, as that of gravity, or the resistance offered by a containing vessel.

The force with which a gas or vapor tends to expand is called its **tension,** or its **elastic force.** When the pressure exerted by a gas or vapor on a surface is uniform, that is, when it is the same at all points of the surface, we take as its unit of measure the pressure on *a square inch* of the surface. If we denote this unit by p, the area pressed by a, and the total pressure by P, we then have,

$$P = ap \quad . \; . \; . \; . \; . \quad (326)$$

When the pressure is variable, we take for the unit of measure at any point the pressure that would be exerted on a square inch, if the pressure were the same at every point of the square inch, as at the point in question. But we may regard the pressure on any infinitesimal element, dm, as constant throughout the element; hence, in this case we have, as before,

$$P = dm \times p \quad . \; . \; . \; . \; . \quad (327)$$

Many of the principles already demonstrated for liquids hold good for gases and vapors, but there are certain properties arising from elasticity which are peculiar to aëriform bodies, some of which it is now proposed to investigate.

Atmospheric Air.

192. The gaseous fluid that envelops our globe, and extends on all sides to a distance of many miles, is called the **atmosphere**. It consists principally of nitrogen and oxygen, together with small but variable portions of watery vapor and carbonic acid, all in a state of mixture. On an average, it is found that 1000 parts by volume of atmospheric air, taken near the surface of the earth, consist of about,

788 parts of nitrogen,
197 parts of oxygen,
14 parts of watery vapor,
1 part of carbonic acid.

The atmosphere may be taken as a type of gases, for it is found by experiment that the laws regulating density, expansibility, and elasticity, are the same for all gases and vapors, so long as they maintain a purely gaseous form. It is found, however, in the case of vapors, and of those gases which have been reduced to a liquid form, that the laws change just before actual liquefaction.

This change appears to be somewhat analogous to that observed when water passes from the liquid to the solid form. Although water does not actually freeze till reduced to a temperature of 32° Fah., it is found that it reaches its maximum density at about 39°, at which temperature the particles seem to commence arranging themselves according to some new law, preparatory to taking on the solid form.

Atmospheric Pressure.

193. If a tube, 35 or 36 inches long, open at one end and closed at the other, be filled with pure mercury, and inverted in a basin of the same, it is observed that the mercury

at the level of the sea will fall in the tube until the vertical distance from the surface of the mercury in the tube to that in the basin is about 30 inches. This column of mercury is sustained by the pressure of the atmosphere exerted upon the surface of the mercury in the basin, and transmitted through the fluid, according to the general law of *transmission of pressures.* The column of mercury sustained by the elasticity of the atmosphere is called the **barometric column,** because it is generally measured by an instrument called a barometer. In fact, the instrument just described, when provided with a suitable scale for measuring the altitude of the column, is a barometer. The height of the barometric column fluctuates somewhat, even at the same place, on account of changes of temperature, and other causes yet to be considered.

Fig. 153.

Observation has shown, that the average height of the barometric column at the level of the sea, is a little less than 30 inches. But the weight of a column of mercury 30 inches in height is nearly 15 *lbs.*; hence, a pressure of 15 *lbs.* on a square inch is adopted as a unit of measure, and is technically called **an atmosphere.**

This unit is often employed in measuring the pressure of elastic fluids, particularly in the case of steam. Thus, when we say that the pressure of steam in a boiler is two atmospheres, we are to understand that there is a pressure of 30 *lbs.* on each square inch of the interior of the boiler. In general, when we say that the tension of a gas or vapor is equal to n atmospheres, we mean that it is capable of exerting a pressure of n times 15 *lbs.* on each square inch of the surface with which it may be in contact.

The specific gravity of mercury being nearly 13.6, a mercurial column of 30 inches is equivalent to a water column of 34 feet.

Mariotte's Law.

194. When a given mass of any gas or vapor is compressed so as to occupy a smaller space, other things being equal, its elastic force is increased; on the contrary, if its volume is increased, its elastic force is diminished.

The law of increase and diminution of elastic force, first discovered by MARIOTTE, and bearing his name, may be enunciated as follows:

The elastic force of a given mass of any gas, whose temperature remains the same, varies inversely as the volume which it occupies.

As long as the mass remains the same, the density must vary inversely as the volume occupied. Hence, from MARIOTTE's law, it follows, that,

The elastic force of any gas, whose temperature remains the same, varies as its density, and conversely, *the density varies as the elastic force.*

MARIOTTE's law may be verified for atmospheric air, by an instrument called MARIOTTE's tube. This is a tube, $ABCD$, of uniform bore, bent so that its two branches are parallel to each other. The shorter branch, AB, is closed at its upper extremity, whilst the longer one is open. Between the two branches, and attached to the frame, is a scale of equal parts.

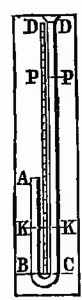

Fig. 153.

To use the instrument, place it in a vertical position, and pour mercury into the tube, until it just cuts off communication between the two branches. The mercury will then stand at the same level, BC, in both branches, and the tension of the air in AB, will be exactly equal to that of the external atmosphere. If an additional quantity of mercury be poured into the longer branch, the air in the shorter branch will be compressed, and the mercury will rise

in both branches, but higher in the longer, than in the shorter one. Suppose the mercury to have risen in the shorter branch, to K, and in the longer one, to P. There will be an equilibrium in the mercury lying below the horizontal plane, KK; there will also be an equilibrium between the tension of the air in AK, and the forces which give rise to that tension. These forces are, the pressure of the external atmosphere, transmitted through the mercury, and the weight of a column of mercury whose base is the cross-section of the tube, and whose altitude is PK. If we denote the height of the column of mercury sustained by the pressure of the external atmosphere, by h, the tension of the air in AK, will be measured by the weight of a column of mercury, whose base is the cross-section of the tube, and whose height is $h + PK$. Since the weight is proportional to the height, the tension of the confined air is proportional to $h + PK$.

Now, whatever may be the value of PK, we have, from the assumed law,

$$AK : AB :: h : h + PK;$$

whence,

$$AK = \frac{AB \cdot h}{h + PK} \quad . \quad . \quad . \quad . \quad . \quad (328)$$

If $PK = h$, we have, $AK = \frac{1}{2}AB$; if $PK = 2h$, we have, $AK = \frac{1}{3}AB$; if $PK = nh$, n being any positive number, entire or fractional, we have, $AK = \dfrac{AB}{n + 1}$. Formula (328), deduced from MARIOTTE'S law, was verified by DULONG and ARAGO for all values of n, up to $n = 27$.

The law may also be verified when the pressure is less than an atmosphere, by the following apparatus: AK is a tube of uniform bore, closed at its upper and open at its lower extremity; CD is a deep cistern of mercury. The tube, AK, is either graduated into equal parts, commencing at A, or has attached to it a graduated scale of brass or ivory.

To use the instrument, pour mercury into the tube till it is nearly full; place the finger over the open end, invert it in the cistern, and depress it till the mercury stands at the same level without and within the tube, and suppose the surface of the mercury in this case to be at B. Then will the tension of the air, in AB, be equal to that of the external atmosphere. If the tube be raised vertically, the air in AB will expand, its tension will diminish, and the mercury will fall in the tube, to maintain the equilibrium. Suppose the level of the mercury in the tube, to have reached K. In this position of the instrument the tension of the air in AK, added to the weight of the column of mercury, KE, will be equal to the tension of the external air.

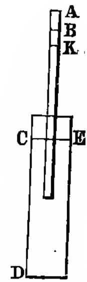

Fig. 154.

Now, it is found, whatever may be the value of KE, that

$$AK : AB :: h : h - EK;$$

whence,

$$AK = \frac{AB \cdot h}{h - EK}. \quad . \quad . \quad . \quad . \quad (329)$$

If $EK = \frac{1}{2}h$, we have, $AK = 2AB$; if $EK = \frac{2}{3}h$, we have $AK = 3AB$; in general, if $EK = \frac{n}{n+1}h$, we have, $AK = AB(n+1)$.

Formula (329) has been verified, for all values of n, up to $n = 111$.

It is a law of Physics that, when a gas is suddenly compressed, heat is evolved, and when a gas is suddenly expanded, heat is absorbed; hence, in making the experiment, care must be taken that the temperature be kept uniform.

More recent experiments have shown that Mariotte's law is not *strictly true*, especially for high tensions, yet its variation is so small that the error committed in regarding it as true is not appreciable in practical mechanics.

Gay Lussac's Law.

195. If the volume of any gas or vapor remain the same, and its temperature be increased, its tension is increased also. If the pressure remain the same, the volume of the gas increases as the temperature is raised.

Let us assume that a given mass of gas or vapor occupies a certain volume at the temperature 32° F.; then the law discovered by GAY LUSSAC, and which bears his name, may be enunciated as follows:

If the volume remains the same, its increase of tension varies as its increase of temperature; if its tension remains the same, its increase of volume varies as its increase of temperature.

According to REGNAULT, if a given mass of air be heated from 32° Fahrenheit to 212°, the tension remaining constant, its volume will be increased by the .3665th part of its volume at 32°. Hence, the increase for each degree of temperature is the .00204th part of its volume at 32°. If we denote the volume at 32° by v, and the volume at the temperature t' degrees, by v', we have,

$$v' = v\,[1 + .00204\,(t' - 32)] \quad . \; . \; . \; . \; . \quad (330)$$

Solving with reference to v, we have,

$$v = \frac{v'}{1 + .00204\,(t' - 32)} \quad . \; . \; . \; . \; . \quad (331)$$

Formula (331) enables us to compute the volume of a mass of air at 32°, when we know its volume at the temperature t' degrees, the pressure remaining constant.

To find the volume at the temperature t'' degrees, we have simply to substitute t'' for t' in (330). Denoting this volume by v'', we have,

$$v'' = v\,[1 + .00204\,(t'' - 32)].$$

Substituting for v its value, from (331), we get,

$$v'' = v' \frac{1 + .00204\,(t'' - 32)}{1 + .00204\,(t' - 32)} \quad . \; . \; . \; . \; . \quad (332)$$

This formula enables us to compute the volume of a mass of air, at a temperature t'', when we know its volume at the temperature t'; and, since the density varies inversely as the volume, we may also, by means of the same formula, find the density of any mass of air, at the temperature t'', when we have given its density at the temperature t'.

Absolute Temperature.

196. In the practical applications of the laws of MARIOTTE and GAY LUSSAC it is often found convenient to reckon temperatures from a *zero point*, so taken that the volume of a gas or vapor of given tension shall always be proportional to its temperature.

To find the position of such a *zero point* on the Fahrenheit scale, let us consider the case of an ideal **air thermometer.** Let AE be a tube of uniform cross-section closed at the bottom and open at the top; let P be an air-tight piston without weight, which moves freely and without friction up and down the tube; and suppose a given mass of air to be confined between the bottom of the tube and the piston.

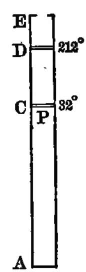

Fig. 155.

When the temperature of the confined air is 32° F. suppose the piston to be at C, and when its temperature is 212° suppose the piston to be at D. Divide CD into 180 equal parts and continue the scale to the bottom of the tube; then if the scale thus determined is numbered from the bottom upward, the lower division being numbered 0, it is obvious that the numbers thus found will be proportional to the corresponding volumes

of the confined air. To find the number of divisions of the scale between C and A. Denote AC by 1; then, from the law of expansion as deduced by REGNAULT, CD will be equal to .3665; and if we denote the number of divisions in CA by x, we shall have,

$$CD : AC :: 180 : x; \quad \text{or} \quad .3665 : 1 :: 180 : x;$$

whence,

$$x = 491, \text{ nearly} \quad . \quad . \quad . \quad . \quad . \quad (333)$$

If, therefore, the Fahrenheit scale be extended downward 491 divisions below the freezing point of water, and if the lowest point of the scale be taken as the zero point, the volume of the confined air will always be proportional to its temperature. On the scale thus determined the freezing point of water is at 491°, and the boiling point is 671°; at the former temperature the volume of the confined air is 1, and at the latter temperature it is 1.3665; and we obviously have,

$$491° : 671° :: 1 : 1.3665.$$

Temperatures counted from the zero point thus determined are called **absolute temperatures.**

The zero point of the **absolute scale** is 491° — 32°, or 459° below the zero point of the Fahrenheit scale; hence, we may convert any ordinary temperature into the corresponding absolute temperature by adding to it 459°. Thus, 25° F. (ordinary temperature) is equivalent to 484° F. absolute temperature.

The absolute zero point of the centigrade scale is 491° ÷ 1.8, or, 273°, nearly, below the centigrade zero point. Hence, any ordinary temperature, as given by the centigrade thermometer, may be converted into absolute centigrade temperature by adding to it 273°. Thus, 15° C. (ordinary temperature) is equivalent to 288° C. absolute temperature.

If we denote the volume of a given mass of gas or vapor by

V, the pressure per square inch on its surface by P, and its absolute temperature by T, the relation between these quantities may be expressed by the equation,

$$\frac{PV}{T} = a\ constant \quad . \quad . \quad . \quad . \quad . \quad (334)$$

If T is constant, we see that P varies inversely as V; which is MARIOTTE'S law.

If V is constant, P varies as T; if P is constant, V varies as T; which is GAY LUSSAC'S law.

Hence, we see that MARIOTTE'S and GUY LUSSAC'S law are both particular cases of a more general law which is given by equation (334).

Manometers.

197. A **manometer** is an instrument for measuring the tension of gases and vapors, particularly of steam. Two principal varieties of manometers are used for measuring the tension of steam, the **open**, and the **closed manometer**.

The Open Manometer.

198. The **open manometer** consists of an open glass tube, AB, terminating near the bottom of a cistern EF. The cistern is of wrought-iron, steam-tight, and filled with mercury. Its dimensions are such, that the upper surface of the mercury will not be materially lowered, when a portion of the mercury is forced up the tube. ED is a tube, by means of which, steam may be admitted from the boiler to the surface of the mercury in the cistern. This tube is sometimes filled with water, through which the pressure of the steam is transmitted to the mercury.

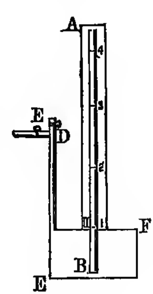

Fig. 156.

To graduate the instrument. All communuication with the boiler is cut off, by closing the stop-cock, E, and communication with the external air is made by opening the stop-cock, D. The point H of the tube at which the mercury stands is then marked 1. From the point H, distances equal to 30, 60, 90, etc., inches are laid off upward, and the corresponding points numbered 2, 3, 4, etc. These divisions correspond to atmospheres, and may be subdivided into tenths and hundredths.

To use the instrument, the stop-cock, D, is closed, and communication made with the boiler, by opening the stop-cock, E. The height to which the mercury rises in the tube indicates the tension of the steam in the boiler, which may be read from the scale in terms of atmospheres and decimals of an atmosphere. If the pressure in pounds is wished, it may at once be found by multiplying the reading of the instrument by 15.

The pressure, when the surface of the mercury stands at 1 is the actual pressure of the atmosphere at the time of observation ; hence, if great accuracy is required, the reading of the instrument must be corrected for the excess of 30 *in.* over the barometric column. Thus, if the reading of the manometer is 3 and the barometric column is 29 *in.*, the true reading is $3 - \frac{1}{30} = 2\frac{29}{30}$.

The principal objection to this kind of manometer is its want of portability, and the great length of tube required, when high tensions are to be measured.

The Closed Manometer.

199. The general construction of the **closed manometer** is the same as that of the open manometer, except that the tube, AB, is closed at the top. The air confined in the tube, is compressed in the same way as in MARIOTTE's tube.

To graduate this instrument. Find by experiment the

point H, that is, the point at which the mercury stands in the tube when the pressure is an atmosphere of 30 inches; this we mark 1, and then we determine the other divisions by means of the following formula.

Denote the distance in inches, from H to the top of the tube, by l; the pressure on the mercury, in atmospheres, by n, and the distance in inches, from H to the upper surface of the mercury in the tube, by x.

The tension of the air in the tube is equal to that on the mercury in the cistern, diminished by the weight of a column of mercury whose altitude is x. Hence, in atmospheres, it is

$$n - \frac{x}{30}.$$

The bore of the tube being uniform, the volume of the compressed air is proportional to its height. When the pressure is 1 atmosphere, the height is l; when the pressure is $\left(n - \frac{x}{30}\right)$ atmospheres, the height is $l - x$. Hence, from MARIOTTE'S law,

$$1 : n - \frac{x}{30} :: l - x : l.$$

Whence, by reduction,

$$x^2 - (30n + l)x = -30l(n - 1).$$

Solving, with respect to x, we have,

$$x = \frac{30n + l}{2} \pm \sqrt{-30l(n-1) + \left(\frac{30n + l}{2}\right)^2}.$$

The upper sign of the radical is not used, as it would give a value for x, greater than l. Taking the lower sign, and assuming $l = 30$ *in.*, we have,

$$x = 15n + 15 - \sqrt{-900(n-1) + (15n + 15)^2} \quad . \quad . \quad . \quad (335)$$

Making $n = 2, 3, 4$, etc., in succession, we find for x, the values, 11.46 *in.*, 17.58 *in.*, 20.92 *in.*, etc. These distances being set off from H, upward, and marked 2, 3, 4, etc., indicate atmospheres. The intermediate spaces may be subdivided by the same formula.

In making the graduation, we have supposed the temperature to remain the same. If, however, it does not remain the same, the reading of the instrument must be corrected by means of a table computed for the purpose.

The instrument is used in the same manner as that already described. Neither can be used for measuring tensions less than 1 atmosphere.

The Siphon Gauge.

200. The siphon gauge is used to measure tensions of gases and vapors, less than an atmosphere. It consists of a tube, ABC, bent so that its two branches are parallel. The branch, BC, is closed at the top, and filled with mercury, which is retained by the pressure of the atmosphere; the branch, AB, is open at the top. If the air be rarefied in any manner, or, if the mouth of the tube be exposed to the action of a gas whose tension is sufficiently small, the mercury will no longer be supported in BC, but will fall in it, and rise in BA. The distance between the surfaces of the mercury in the two branches, given by a scale between them, indicates the tension of the gas. If this distance is expressed in inches, the tension can be found, in atmospheres, by dividing by 30, or, in pounds, by dividing by 2.

Fig. 157.

The Diving-Bell.

201. The **diving-bell** is a bell-shaped vessel, open at the bottom, used for descending into the water. The bell is

placed with its mouth horizontal, and let down by a rope, AB, the whole apparatus being sunk by weights properly adjusted. The air contained in the bell is compressed by the pressure of the water, but its increased elasticity prevents the water from rising to the top of the bell, which is provided with seats for the accommodation of those within the bell. The air, constantly contaminated by breathing, is continually replaced by fresh air, pumped in through a tube, FG. Were there no additional air introduced, the volume of the compressed air, at any depth, might be computed by MARIOTTE'S law. The unit of the compressing force, in this case, is the weight of a column of water whose cross-section is a square inch, and whose height is the distance from DC to the surface of the water.

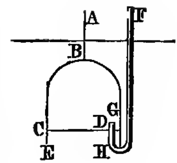

Fig. 158.

The principle of the diving-bell is used in the **diving-dress.** The diver is surrounded by a water-tight envelope, fitted with a helmet, into which air is pumped from above, as in the diving-bell. There is, as in the diving-bell, an escape valve, by means of which circulation of the air is maintained. The diving-bell is used in constructing foundations and other submarine works; the diving-dress is principally used in submarine explorations.

The Barometer.

202. The **barometer** is an instrument for measuring the pressure of the atmosphere. It consists of a glass tube, hermetically sealed at one extremity, filled with mercury, and inverted in a basin of that fluid. The pressure of the air is indicated by the height of mercury that it supports.

A variety of forms of mercurial barometers have been devised, all involving the same mechanical principle. The most important of these are the **siphon** and the **cistern barometers.**

The Siphon Barometer.

203. The **siphon barometer** consists of a tube, *CDE*, bent so that its two branches, *CD* and *DE*, are parallel to each other. A scale is placed between them, and attached to the same frame as the tube. The longer branch, *CD*, is hermetically sealed at the top, and filled with mercury; the shorter one is open to the air. When the instrument is placed vertically, the mercury sinks in the longer branch and rises in the shorter one. The distance between the surface of the mercury in the two branches indicates the pressure of the atmosphere. To prevent shocks when handling the instrument, the tube is drawn to a narrow neck in the neighborhood of the point marked *B* in the diagram.

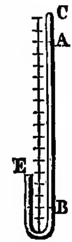

Fig. 159.

The Cistern Barometer.

204. The **cistern barometer** consists of a glass tube, filled with mercury, and inverted in a cistern of the same. The tube is surrounded by a frame of metal, attached to the cistern. Two longitudinal openings, near the upper part of the frame, permit the upper surface of the mercury to be seen. A slide, moved up and down by a rack and pinion, may be brought exactly to the upper level of the mercury. The height of the column is then read from a scale, whose 0 is at the surface of the mercury in the cistern. The scale is graduated to inches and tenths, and smaller divisions are read by means of a vernier.

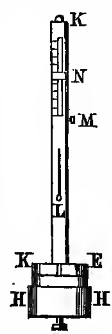

Fig. 160.

The figure shows the essential parts of a complete cistern barometer; *KK* represents the frame; *HH*, the cistern, the upper part of which is of glass, that the mercury in the cistern may be seen through it; *L*, a thermometer, to show the temperature of

the mercury; *N*, a sliding-ring bearing the vernier, and moved up and down by the pinion, *M*.

The cistern is shown on an enlarged scale in Fig. 161; *A* is the barometer tube, terminating in a small opening, to prevent sudden shocks when the instrument is moved from place to place; *H*, the frame of the cistern; *B*, the upper portion of the cistern, made of glass, that the mercury may be seen; *E*, a piece of ivory, projecting from the upper surface of the cistern, whose point corresponds to the 0 of the scale; *CC*, the lower part of the cistern, made of leather, or other flexible material, and firmly attached to the glass part; *D*, a screw, working through the frame, and against the bottom of the bag, *CC*, by means of a plate, *P*. The screw, *D*, serves to bring the surface of the mercury to the point of ivory, *E*, and also to force the mercury to the top of the tube, when it is desired to transport the barometer from place to place.

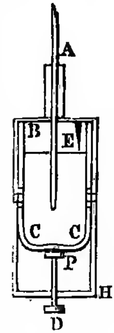

Fig. 161.

To use this barometer, it is suspended vertically, and the level of the mercury in the cistern brought to the point of ivory, *E*, by the screw, *D*; a smart rap on the frame will detach the mercury from the glass, to which it tends to adhere. The ring, *N*, is run up or down till its lower edge appears tangent to the surface of the mercury in the tube, and the altitude is read from the scale. The height of the attached thermometer should also be noted.

The requirements of a good barometer are, sufficient width of tube, perfect purity of mercury, and a scale with an accurately graduated vernier.

The bore of the tube should be as large as practicable, to diminish the effect of capillary action. On account of the repulsion between the glass and mercury, the latter is depressed in the tube, and this depression increases as the diameter of the tube diminishes.

In all cases, this depression should be allowed for, and the reading corrected by a table computed for the purpose.

To secure purity of the mercury, it should be carefully distilled, and after the tube is filled, it should be boiled to drive off any bubbles of air that may adhere to the tube.

Uses of the Barometer.

205. The primary object of the barometer is, to measure the pressure of the atmosphere. It is used by mariners as a weather-glass. It is also employed for determining the heights of points on the earth's surface, above the level of the sea. The principle on which it is employed for the latter purpose is, that the pressure of the atmosphere at any place depends on the weight of a column of air reaching from the place to the upper limit of the atmosphere. As we ascend above the level of the ocean, the weight of the column diminishes; consequently, the pressure becomes less, a fact which is shown by the mercury falling in the tube.

Difference of Level Between Two Stations.

206. Let us suppose the atmosphere to be in a state of equilibrium, and the temperature of the atmosphere and of the mercury to be 32° F., both at the upper and at the lower station; and furthermore, suppose that the force of gravtity does not vary in passing from one station to the other.

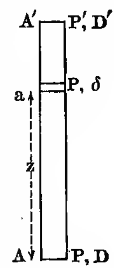

Fig. 162.

Conceive a cylindrical column of air whose cross-section is 1 square inch to extend from the bottom of the atmosphere to the top. Then, because the atmosphere is supposed to be in equilibrium, the lateral pressures in each horizontal layer will balance each other, and we may treat the column as though it were inclosed in a vertical tube.

Let AA' be a portion of such a column, A being in the horizontal plane passing through the lower station, and A' in

the horizontal plane passing through the upper station; denote the pressure of the atmosphere at A by P, and at A' by P'; denote the density of the air at A by D, and at A' by D'; at any point a between A and A' and at a distance from A equal to z, let the pressure be denoted by p and the density by δ.

The change of pressure in passing from the height z to the height $z + dz$, denoted by dp, will obviously be equal to the weight of the infinitesimal layer whose height is dz; but the weight of this layer, equal to its volume *into* its density *into* gravity, is equal to $dz \times \delta \times g$; and because p decreases as z increases, we have,

$$dp = -dz \times \delta \times g \quad . \quad . \quad . \quad . \quad . \quad (336)$$

From Mariotte's law we have,

$$p : P :: \delta : D, \quad \text{or,} \quad p = \frac{P}{D}\delta \quad . \quad . \quad . \quad . \quad . \quad (337)$$

Dividing (336) by (337), we find,

$$\frac{dp}{p} = -\frac{Dg}{P}dz. \quad . \quad . \quad . \quad . \quad . (338)$$

Integrating (338) from the bottom of the column to the top, and denoting the height AA' by h, we have,

$$lP' - lP = -\frac{Dg}{P}h, \quad \text{or,} \quad lP - lP' = \frac{Dg}{P}h, \quad . \quad . \quad . \quad (339)$$

in which l denotes *the Naperian logarithm.* Multiplying both members of (339) by M, the modulus of the common system, and reducing, we have,

$$log\left(\frac{P}{P'}\right) = \frac{MDg}{P}h,$$

and solving, we find,

$$h = \frac{P}{MDg} \log \left(\frac{P}{P'}\right), \quad . \quad . \quad . \quad . \quad . \quad (340)$$

in which *log* denotes *the common logarithm.*

For the case assumed, the pressures at A and A' are measured by the heights of the mercurial columns at these stations; denoting these heights by H and H', we have finally,

$$h = \frac{P}{MDg} \log \left(\frac{H}{H'}\right), \quad . \quad . \quad . \quad . \quad . \quad (341)$$

which is the **fundamental barometrical formula.** In applying formula (341) to practice, a great number of corrections must be made. It must be corrected for the temperature of the stratum of air between the stations A and A'; it must be corrected for difference of temperature of the columns of mercury at the two stations; it must be corrected for the variation of gravity between the level of the sea and each of the two stations; and it must be corrected for the variation in the amount of aqueous vapor at the two stations. The method of making these corrections does not fall within the scope of this treatise.

For the complete formula, as used in geodesy, the reader is referred to the exhaustive treatise, "On the Use of the Barometer in Surveys and Reconnaissances," by Colonel R. S. WILLIAMSON, U. S. Engineers, published by the Engineer Department for the use of the army engineers; and also to the admirable tables of GUYOT, published by the Smithsonian Institution. In both of these works elaborate tables are given, by means of which the process of *barometrical leveling* is made comparatively simple. In "Professional Papers, No. 12," published for the use of the Corps of U. S. Engineers, pp. 146–180, may be found PLANTAMOUR'S modification of BESSEL'S formula, arranged and adapted to English measures, with a complete set of tables for its use, by Colonel WILLIAMSON.

Work Due to the Expansion of a Gas or Vapor.

207. Let the gas or vapor be confined in a cylinder closed at is lower end, and having a piston working air-tight. When the gas occupies a portion of the cylinder whose height is h, denote the pressure on each square inch of the piston by p; when the gas expands, so that the altitude of the column becomes x, denote the pressure on a square inch by p'.

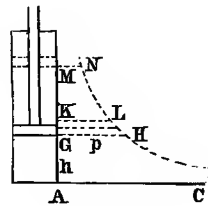

Fig. 168.

Since the volumes of the gas, under these suppositions, are proportional to their altitudes, we shall have, from MARIOTTE'S law,

$$p : p' :: x : h;$$

whence

$$xp' = ph \quad . \quad . \quad . \quad . \quad . \quad (342)$$

If we suppose p and h to be constant, and x and p' to vary, the above equation will be that of an equilateral hyperbola, whose asymptotes are AC and AM.

From (342) we have,

$$p' = \frac{ph}{x}.$$

The elementary quantity of work performed on each square inch of the piston whilst the gas is expanding from the height x to the height $x + dx$ is equal to $p'\,dx$; denoting this by dQ', we have, after substituting for p' its value from the preceding equation,

$$dQ' = ph\,\frac{dx}{x}.$$

Integrating between the limits h and x, we have,

$$Q' = ph\ (lx - lh) = ph \cdot l\left(\frac{x}{h}\right), \quad . \quad . \quad . \quad . \quad . \quad (343)$$

in which l denotes the Naperian logarithm.

In (343), Q' is the quantity of work performed on each square inch of the piston; if we denote the area of the piston in square inches by A and the total quantity of work by Q, we have, after substituting for $\frac{x}{h}$ its value taken from (342),

$$Q = Aph \cdot l\left(\frac{p}{p'}\right) \quad . \quad . \quad . \quad . \quad . \quad (344)$$

If we denote by c the number of cubic feet of gas when the pressure on a square inch of the piston is p, and suppose it to expand till the pressure is p', we shall have $c = Ah$, or, if A is expressed in square feet, we shall have $c = \frac{Ah}{144}$, or $h = \frac{144c}{A}$. This, in (344), gives

$$Q = 144cp \cdot l\left(\frac{p}{p'}\right), \quad . \quad . \quad . \quad . \quad . \quad (345)$$

which gives the quantity of work of c cubic feet of gas whilst expanding from a pressure of p pounds on a square inch to a pressure of p' pounds.

Steam.

208. If water be exposed to the atmosphere, at ordinary temperatures, a portion is converted into vapor, which mixes with the atmosphere, constituting one of the permanent elements of the aërial ocean. The tension of watery vapor thus formed, is very slight, and the atmosphere soon ceases to absorb any more. If the temperature of the water be raised, an additional amount of vapor is evolved, and of greater tension. When the temperature is raised to that point at which the tension of the vapor is equal to that of the atmos-

phere, ebullition commences, and the vaporization goes on with great rapidity. If heat be added beyond the point of ebullition, neither the water nor the vapor will increase in temperature till all of the water is converted into steam.

When water is converted into steam under a pressure of one atmosphere, each cubic inch is expanded into about 1700 cubic inches of steam, of the temperature of 212°; or, since a cubic foot contains 1728 cubic inches, we may say, in round numbers, that **a cubic inch of water is converted into a cubic foot of steam.**

If water is converted into steam under a greater or less pressure than one atmosphere, the density will be increased or diminished, and, consequently, the volume will be diminished or increased. The temperature being also increased or diminished, the increase of density or decrease of volume will not be exactly proportional to the increase of pressure; but, for purposes of approximation, we may consider the densities as directly and the volumes as inversely proportional to the pressures under which the steam is generated. Under this hypothesis, if a cubic inch of water be evaporated under a pressure of a half atmosphere, it will afford two cubic feet of steam; if generated under a pressure of two atmospheres, it will only afford a half cubic foot of steam.

Work of Steam.

209. When water is converted into steam, a certain amount of work is generated, and, from what has been shown, this amount of work is very nearly the same, whatever may be the temperature at which the water is evaporated.

Suppose a cylinder, whose cross-section is one square inch, to contain a cubic inch of water, above which is an air-tight piston, that may be loaded with weights at pleasure. In the first place, if the piston is pressed down by a weight of 15 pounds, and the inch of water converted into steam, the

weight will be raised to the height of 1728 inches, or 144 feet. Hence, the quantity of work is 144×15, or, 2160 units. Again, if the piston be loaded with a weight of 30 pounds, the conversion of water into steam will give but 864 cubic inches, and the weight will be raised through 72 feet. In this case, the quantity of work will be 72×30, or 2160 units, as before. We conclude, therefore, that the quantity of work is the same, or nearly so, whatever may be the pressure under which the steam is generated. We also conclude, that the quantity of work is nearly proportional to the fuel consumed.

Besides the quantity of work developed by simply converting an amount of water into steam, a further quantity of work is developed by allowing the steam to expand after entering the cylinder. This principle is made use of in steam-engines working expansively.

To find the quantity of work developed by steam acting expansively. Let AB represent a cylinder, closed at A, and having an air-tight piston B. Suppose the steam to enter at the bottom of the cylinder, and to push the piston upward to C, and then suppose the opening at which the steam enters to be closed. If the piston is not too heavily loaded, the steam will continue to expand, and the piston will be raised to some position, B. The expansive force of the steam will obey MARIOTTE'S law, and the quantity of work due to expansion will be given by equation (344).

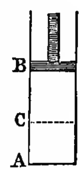

Fig. 164.

Denote the area of the piston in square inches, by A; the pressure of the steam on each square inch, up to the moment when the communication is cut off, by p; the distance AC, through which the piston moves before the steam is cut off, by h; and the distance AB, by nh.

If we denote the pressure on each square inch, when the piston arrives at B, by p', we shall have, by MARIOTTE'S law,

$$p : p' :: nh : h, \quad \therefore p' = \frac{p}{n},$$

an expression which gives the limiting value of the load of the piston.

The quantity of work due to expansion being denoted by Q', we shall have, from equation (344),

$$Q' = Aph \times l\left(\frac{nh}{h}\right) = Aph \cdot l(n) \quad . \quad . \quad . \quad . \quad . \quad (346)$$

If we denote the quantity of work of the steam, whilst the piston is rising to C, by Q'', we shall have,

$$Q'' = Aph.$$

Denoting the total quantity of work during the entire stroke of the piston, by Q, we shall have,

$$Q' + Q'' = Q = Aph\,[1 + l(n)] \quad . \quad . \quad . \quad . \quad . \quad (347)$$

Efflux of a Gas or Vapor.

210. Suppose the gas to escape from a small orifice, and denote its velocity by v. Denote the weight of a cubic foot of the gas by w, and the number of cubic feet discharged in one second by c, then will the mass escaping in one second, be equal to $\frac{cw}{g}$, and its kinetic energy will be equal to $\frac{cw}{2g}v^2$. But, from Art. 148, the kinetic energy is equal to the accumulated work. If, therefore, we denote the accumulated work by Q, we shall have,

$$Q = \frac{cw}{2g}v^2.$$

But the accumulated work is due to the expansion of the

gas, and if we denote the pressure within the orifice by p, and without, by p', we shall have, from Art. 207,

$$Q = 144cp \times l\left(\frac{p}{p'}\right).$$

Equating the second members, we have,

$$\frac{cw}{2g}v^2 = 144cp \times l\left(\frac{p}{p'}\right);$$

whence,

$$v = 12\sqrt{\frac{2gp}{w} \times l\left(\frac{p}{p'}\right)}.$$

Substituting for g, its value, $32\frac{1}{6}$ *ft.*, we have, after reduction,

$$v = 96\sqrt{\frac{p}{w} \times l\left(\frac{p}{p'}\right)} \quad . \quad . \quad . \quad . \quad . \quad (348)$$

When the difference between p and p' is small, the preceding formula can be simplified.

Since $\frac{p}{p'} = 1 + \frac{p - p'}{p'}$, we have, from the logarithmic formula,

$$l\left(\frac{p}{p'}\right) = l\left(1 + \frac{p - p'}{p'}\right) = \frac{p - p'}{p'} - \frac{1}{2}\left(\frac{p - p'}{p'}\right)^2 + \text{etc.}$$

When $p - p'$ is very small, the second, and all succeeding terms of the development, may be neglected, in comparison with the first term. Hence,

$$l\left(\frac{p}{p'}\right) = \frac{p - p'}{p'}.$$

Substituting, in the formula above deduced, we have,

$$v = 96\sqrt{\frac{p}{w} \times \frac{p - p'}{p'}};$$

or, since $\frac{p}{p'}$ is, under the supposition just made, equal to 1, we have, finally,

$$v = 96\sqrt{\frac{p - p'}{w}} \quad \ldots\ldots \quad (349)$$

Coefficient of Efflux.

211. When air issues from an orifice, the section of the current undergoes a change of form, analogous to the contraction of the vein in liquids, and for similar reasons. If we denote the coefficient of efflux by k, the area of the orifice by A, and the quantity of air delivered in n seconds by Q, we shall have, from equation (348),

$$Q = 96knA\sqrt{\frac{p}{w}\, l\left(\frac{p}{p'}\right)} \quad \ldots\ldots \quad (350)$$

In which w is to be expressed in the same unit as p and p'. According to Koch, the value of k is equal to .58 when the orifice is in a thin plate; equal to .74 when the gas issues through a tube 6 times as long as it is wide; and equal to .85 when it issues through a conical nozzle 5 times as long as the diameter of the orifice, and whose elements make an angle of 6° with the axis.

X.—HYDRAULIC AND PNEUMATIC MACHINES.

Definitions.

212. Hydraulic machines are machines for raising and distributing water, as *pumps, siphons, hydraulic rams,* and the like. The name is also applied to machines in which water power is the motor, or in which water is employed to transmit pressures, as *water-wheels, hydraulic presses,* and the like.

Pneumatic machines are machines to rarefy and condense air, or to impart motion to air, as *air-pumps, ventilating blowers,* and the like. The name is also applied to those machines in which the kinetic energy of air is the motive power, such as *windwills,* and the like.

Water Pumps.

213. A **water pump** is a machine for raising water from a lower to a higher level, by the aid of atmospheric pressure. Three separate principles are employed in pumps: the **sucking**, the **lifting**, and the **forcing** principle. Pumps are named according to the principles employed.

Sucking and Lifting Pump.

214. This pump consists of a **barrel**, A, to the lower extremity of which is attached a **sucking-pipe**, B, leading to a reservoir. An air-tight piston, C, is worked up and down in the barrel by a lever, E, attached to a piston-rod, D; P is a valve opening upward, which, when the pump is at rest, closes by its own weight. This valve is called the **piston-**

valve. A second valve, *G,* also opening upward, is placed at the junction of the pipe with the barrel; this is called the **sleeping-valve.** The space, *LM,* through which the piston moves up and down, is **the play of the piston.**

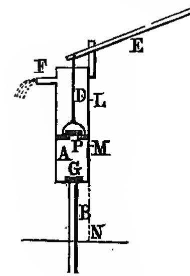

Fig. 165.

To explain the action of the pump: suppose the piston to be in its lowest position, and everything in equilibrium. If the extremity of the lever, *E,* be depressed, and the piston raised, the air in the lower part of the barrel is rarefied, and that in the pipe, *B,* by virtue of its greater tension, opens the valve, and a portion escapes into the barrel. The air in the pipe, thus rarefied, exerts less pressure on the water in the reservoir than the external air, and, consequently, the water rises in the pipe, until the tension of the internal air, plus the weight of the column of water raised, is equal to the tension of the external air; the valve, *G,* then closes by its own weight.

If the piston be again depressed to the lowest limit, the air in the lower part of the barrel is compressed, its tension becomes greater than that of the external air, the valve, *P,* is forced open, and a portion of the air escapes. If the piston be raised once more, the water, for the same reason as before, rises still higher in the pipe, and after a few double strokes of the piston, the air is completely exhausted from beneath the piston, the water passes through the piston-valve, and finally escapes at the spout, *F.*

The water is raised to the piston by the pressure of the air on the surface of the water in the reservoir; hence, the piston should not be placed at a greater distance above the water in the reservoir, than the height at which the pressure of the air will sustain a column of water. In fact, it should be placed a little lower than this limit. The specific gravity of mer-

cury being about 13.5, the height of a column of water that will counterbalance the pressure of the atmosphere may be found by multiplying the height of the barometric column by $13\frac{1}{2}$.

At the level of the sea the average height of the barometric column is $2\frac{1}{2}$ feet; hence, the theoretical height to which water can be raised by the principle of suction alone, is a little less than 34 feet.

The water having passed through the piston-valve, may be raised to any height by the lifting principle, the only limitation being the strength of the pump.

There are certain relations that must exist between the play of the piston and its height above the water in the reservoir, in order that the water may be raised to the piston; if the play is too small, it will happen after a few strokes of the piston, that the air in the barrel is not sufficiently compressed to open the piston-valve; when this state of affairs takes place, the water ceases to rise.

To investigate the relation that should exist between the play and the height of the piston above the water. Denote the play of the piston by p, the distance from the surface of the water in the reservoir to the highest position of the piston by a, and the height at which the water ceases to rise by x. The distance from the water in the pump to the highest position of the piston will be $a - x$, and the distance to the lowest position of the piston $a - p - x$. Denote the height at which the atmospheric pressure sustains a column of water in a vacuum, by h, and the weight of a column of water, whose base is the cross-section of the pump, and altitude is 1, by w; then will wh denote the pressure of the atmosphere exerted upward through the water in the reservoir and pump.

When the piston is at its lowest position, the pressure of the confined air must be equal to that of the external atmosphere; that is, to wh. When the piston is at its highest

position, the confined air will be rarefied, the volume occupied being proportional to its height. Denoting the pressure of this rarefied air by wh', we shall have, from MARIOTTE'S law,

$$wh : wh' :: a - x : a - p - x.$$

$$\therefore \quad h' = h\frac{a - p - x}{a - x}.$$

As the water does not rise when the piston is in its highest position, the pressure of the rarefied air *plus* the weight of the column already raised, will be equal to the pressure of the external atmosphere; or,

$$wh\frac{a - p - x}{a - x} + wx = wh.$$

Solving this with respect to to x, we have,

$$x = \frac{a \pm \sqrt{a^2 - 4ph}}{2} \quad . \quad . \quad . \quad . \quad . \quad (351)$$

If, $\qquad 4ph > a^2; \quad \text{or,} \quad p > \frac{a^2}{4h},$

the value of x is imaginary, and there is no point at which the water ceases to rise. Hence, the above inequality expresses the relation that must exist, in order that the pump may be effective. This condition, expressed in words, gives the following rule:

The play of the piston must be greater than the square of the distance from the water in the reservoir, to the highest position of the piston, divided by four times the height at which the atmosphere will support a column of water in a vacuum.

Let it be required to find the least play of the piston, when its highest position is 16 feet above the water in the reservoir, and the barometer at 28 inches.

In this case,

$$a = 16\,ft., \quad \text{and} \quad h = 28\,in. \times 13\tfrac{1}{2} = 378\,in. = 31\tfrac{1}{2}\,ft.$$

Hence, $$p > \tfrac{256}{126}\,ft.; \quad \text{or,} \quad p > 2\tfrac{2}{63}\,ft.$$

To find the quantity of work required to make a double stroke of the piston, after the water has reached the spout.

In depressing the piston, no force is required, except that necessary to overcome inertia and friction. Neglecting these for the present, the quantity of work in the downward stroke, may be regarded as 0. In raising the piston, its upper surface is pressed downward, by the pressure of the atmosphere, wh, *plus* the weight of the column of water from the piston to the spout; and it is pressed upward, by the pressure of the atmosphere, transmitted through the pump, *minus* the weight of a column of water, whose cross-section is that of the barrel, and whose altitude is the distance from the piston, to the water in the reservoir. If we subtract the latter from the former, the difference will be the downward pressure. This difference is equal to the weight of a column of water, whose base is the cross-section of the barrel, and whose height is the distance of the spout above the reservoir. Denoting this height by H, the pressure is equal to wH. The path through which the pressure is exerted during the ascent of the piston, is the play of the piston, or p. Denoting the quantity of work required, by Q, we shall have,

$$Q = wpH.$$

But wp is the weight of a volume of water, whose base is the cross-section of the barrel, and whose altitude is the play of the piston. Hence, Q is equal to the quantity of work necessary to raise this volume of water from the level of the reservoir to the spout. This volume is evidently equal to that actually delivered at each double stroke of the piston. Hence, the quantity of work expended in pumping, with the sucking

and lifting pump, hurtful resistances being neglected, is equal to the quantity of work necessary to lift the amount of water, actually delivered, from the level of the reservoir to the spout. In addition to this, a sufficient amount of power must be exerted to overcome hurtful resistances.

The disadvantage of this pump, is the irregularity with which the force acts, being 0 in depressing the piston, and a maximum in raising it. This is an important objection when machinery is employed in pumping; but it may be partially overcome, by using two pumps, so arranged, that one piston ascends as the other descends. Another objection to the use of this pump, is the irregularity of flow, the inertia of the column of water having to be overcome at each upward stroke.

Modification of the Lifting Pump.

215. To correct the irregularity of flow, it is customary to attach to the piston-rod a cylindrical piece, *P*, called a **plunger**. This piece, which moves up and down with the piston, is so adjusted as to displace in its descent one half the quantity of water that is delivered at each double stroke of the piston. Then, during the up stroke of the piston, an amount of water will be *lifted* up to the spout equal to that which is *forced* up by the plunger in the down stroke. In this manner the flow is rendered nearly uniform, and the rate of work of the power is made nearly constant.

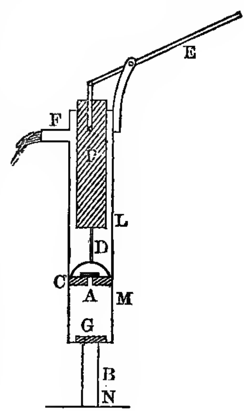

Fig. 166.

Sucking and Forcing Pump.

216. This pump consists of a barrel, *A*, with a sucking pipe, *B*, and a sleeping-valve, *G*, as in the pump just dis-

cussed. The piston, *C*, is solid, and is worked up and down by a lever, *E*, and a piston-rod, *D*. At the bottom of the barrel, a pipe leads to an **air-vessel**, *K*, through a second sleeping-valve, *F*, which opens upward, and closes by its own weight. A **delivery-pipe**, *H*, enters the air-vessel at the top, and terminates near the bottom.

To explain the action of this pump, suppose the piston, *C*, to be in its lowest position. If the piston be raised to its highest position, the air in the barrel is rarefied, its tension is diminished, the air in the tube, *B*, thrusts open the valve, and a portion escapes into the barrel. The pressure of the external air then forces water up the pipe, *B*, until the tension of the rarefied air, *plus* the weight of the water raised, is equal to the tension of the external air. An equilibrium being produced, the valve, *G*, closes by its own weight.

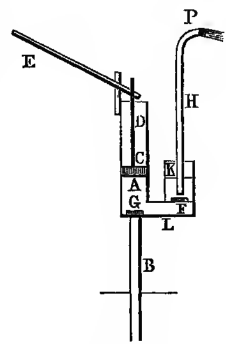

Fig. 167.

If the piston be depressed, the air in the barrel is condensed, its tension increases till it becomes greater than that of the external air, when the valve, *F*, is thrust open, and a portion escapes through the delivery-pipe, *H*. After a few double strokes of the piston, the water rises through the valve, *G*, and, as the piston descends, is forced into the air-vessel, the air is condensed in the upper part of the vessel, and, acting by its elastic force, urges a portion of the water up the delivery-pipe and out at the spout, *P*. The object of the air-vessel is, to keep up a continued stream through the pipe, *H*, otherwise it would be necessary to overcome the inertia of the entire column of water in the pipe at every double stroke. The flow having commenced, a volume of water is delivered from the spout, at each double stroke, equal to that of a cylinder whose

base is the area of the piston, and whose altitude is the play of the piston.

The same condition as to the play of the piston must exist as in the sucking and lifting pump.

To find the quantity of work consumed at each double stroke, after the flow has become regular, hurtful resistance being neglected :

When the piston is descending, it is pressed downward by the tension of the air on its upper surface, and upward by the tension of the atmosphere, transmitted through the delivery-pipe, *plus* the weight of a column of water whose base is the area of the piston, and whose altitude is the distance of the spout above the piston. This distance is variable during the stroke, but its mean value is the distance of the middle of the play below the spout. The difference between these pressures is exerted upward, and is equal to the weight of a column of water whose base is the area of the piston, and whose altitude is the distance from the middle of the play to the spout. The distance through which the force is exerted, is the play of the piston.

Denoting the quantity of work during the descending stroke, by Q', the weight of a column of water, whose base is the area of the piston, and altitude is 1, by w, and the height of the spout above the middle of the play, by h', we have,

$$Q' = wh' \times p.$$

When the piston is ascending, it is pressed downward by the tension of the atmosphere on its upper surface, and upward by the tension of the atmosphere, transmitted through the water in the reservoir and pump, *minus* the weight of a column of water whose base is the area of the piston, and whose altitude is the height of the piston above the reservoir. This height is variable, but its mean value is the height of the middle of the play above the reservoir. The distance through which this force is exerted, is the play of the piston.

Denoting the quantity of work during the ascending stroke, by Q'', and the height of the middle of the play above the reservoir, by h'', we have,

$$Q'' = wh'' \times p.$$

Denoting the entire quantity of work during a double stroke, by Q, we have,

$$Q = Q' + Q'' = wp\,(h' + h'') \quad . \; . \; . \; . \; . \quad (352)$$

But wp is the weight of a volume of water, whose base is the piston, and whose altitude is the play; that is, it is the weight of the volume delivered at each double stroke.

The quantity, $h' + h''$, is the height of the spout above the reservoir. Hence, the work expended is equal to that required to raise the volume delivered from the level of the reservoir to the spout. To this must be added the work necessary to overcome hurtful resistances, as friction, etc.

If $h' = h''$, we have, $Q' = Q''$; that is, the quantity of work during the ascending stroke, is equal to that during the descending stroke. Hence, the work of the motor is more nearly uniform, when the middle of the play is at equal distances from the reservoir and spout.

Forcing Pump with Plunger.

217. A forcing pump may be constructed as shown in Fig. 168. The barrel of the pump, A, is extended into a lateral cylinder, K, in which is a plunger that is worked back and forth through a packing-box, B, by means of the lever, LN. The sucking-pipe, S, and the delivery-pipe, D, are separated from the barrel by sleeping valves, G and F, opening upward and closing by their own weight.

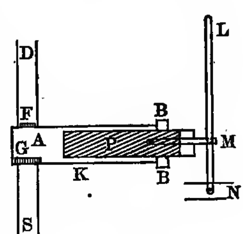

Fig. 168.

The method of filling the pump is as follows: When the plunger is thrust toward A in the figure, the air in the barrel is condensed, forces open the valve, F, and a portion escapes into the delivery-pipe; when the plunger is drawn back the air in the barrel is rarefied, the greater tension of the air in the sucking-pipe opens the valve, G, and the water rises to a certain height in D. A few double strokes fill the pump with water, after which there will be delivered at each double stroke a quantity of water whose volume is equal to that of a cylinder whose diameter is equal to that of the plunger, and whose altitude is equal to the play of the plunger.

In the pump just described there is no air-chamber, and as a consequence the flow is irregular. To prevent *binding*, the lever, LN, is connected with the plunger, P, by means of a short link, having a hinge joint at each extremity. A similar arrangement is made for like purpose in the pumps previously described.

Fire-Engine.

218. The **fire-engine** is a *double* sucking and forcing pump, the piston-rods being so connected, that when one piston ascends, the other descends. The sucking and delivery pipes are made of leather, and attached to the machine by metallic screw-joints.

The figure exhibits a cross-section of the essential parts of an ordinary fire-engine.

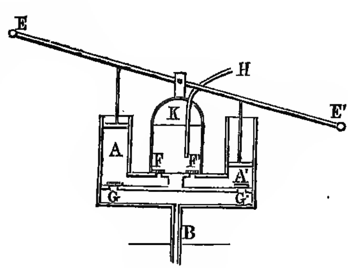

Fig. 169.

A, A', are the barrels, the pistons are connected by rods with the lever, E, E'; B is the sucking-pipe, terminating in a reservoir from which the water may enter

either barrel through the valves, *G*, *G'*; *K* is the air-vessel, common to both pumps, and communicating with them by valves, *F*, *F'*; *H* is the delivery-pipe.

It is mounted on wheels for convenience of locomotion. The lever, *E*, *E'*, is worked by rods at right angles to the lever, so arranged that several men can apply their strength at once. The action of the pump differs in no respect from that of the forcing pump; but when the instrument is worked vigorously, a large quantity of water is forced into the air-vessel, the tension of the air is much augmented, and its elastic force, thus brought into play, propels the water to a considerable distance from the delivery-pipe. It is this capacity of throwing a jet of water to a great distance, that gives to the engine its value for extinguishing fires.

A pump similar to the fire-engine, is often used, under the name of **the double-action forcing-pump,** for other purposes.

The Rotary Pump.

219. The **rotary pump** is a modification of the sucking and forcing pump. Its construction will be understood from the drawing, which represents a section through the axis of the sucking-pipe, at right angles to the axis of the rotating portion.

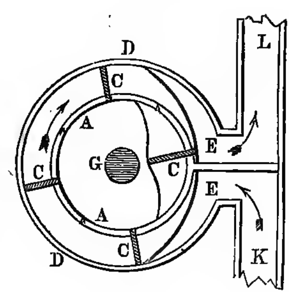

Fig. 170.

A is a ring of metal, revolving about an axis; *D*, *D*, is a second ring of metal, concentric with the first, and forming with it an intermediate annular space. This space communicates with the sucking-pipe, *K*, and the delivery-pipe, *L*. Four radial paddles, *C*, are so disposed as to slide backward and forward through suitable openings in the ring *A*, and are moved around with

it. *G* is a guide, fastened to the end of the cylinder inclosing the revolving apparatus, and cut as represented in the figure ; *E*, *E*, are springs, attached to the ring, *D*, and acting, by their elastic force, to press the paddles firmly against the guide. These springs do not impede the flow of water *from* the pipe, *K*, *into* the pipe, *L*.

When the axis, *O*, revolves, each paddle, as it passes the partition, is pressed against the guide, but is forced out again, by the form of the guide, against the wall, *D*. Each paddle drives the air in front of it in the direction of the arrow-head, and finally expels it through the pipe, *L*. The air behind the paddle is rarefied, and the pressure of the external air forces a column of water up the pipe. After a few revolutions, the air is entirely exhausted from the pipe, *K*. The water enters the channel, *C*, *C*, and is forced up the pipe, *L*, from which it escapes by a spout. The work expended in raising a volume of water to the spout, by this pump, is equal to that required to lift it from the level of the cistern to the spout. This may be shown in the same manner as was explained under the head of the sucking and forcing pump. To this quantity of work, must be added the work necessary to overcome hurtful resistances.

A machine, similar to the rotary pump, is constructed for exhausting foul air from a mine ; or, by reversing the direction of rotation, to force fresh air to the bottom of the mine.

The Hydraulic Press.

220. The **hydraulic press** is a machine for exerting a great pressure through a small space. It is used in compressing seeds to obtain oil, in packing hay and other goods, also in raising great weights. Its construction, though requiring the use of a forcing-pump, depends upon the principle of equal pressures, (Art. 167).

It consists of two cylinders, A and B, each provided with a solid piston. The cylinders communicate by a pipe, C, whose entrance to the larger cylinder is closed by a sleeping valve, E. The smaller cylinder communicates with the reservoir K, by a sucking-pipe H, whose upper extremity is closed by a sleeping valve, D. The piston B, is worked by the lever, G. By raising and depressing the lever G, the water is raised from the reservoir and forced into the cylinder, A; and when the space below the piston, F, is filled, a force is exerted upward, as many times greater than that applied to B, as the area of F is greater than B (Art. 167). This force may be utilized in compressing a body, L, between the piston and the frame of the press.

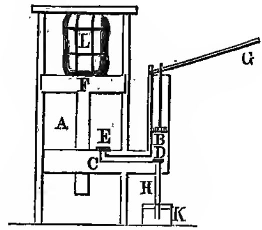

Fig. 171.

Denote the area of the larger piston, by P, of the smaller, by p, the pressure applied to B, by f, and that exerted on F, by F; and we shall have,

$$F : f :: P : p, \quad \therefore F = \frac{fP}{p} \quad \ldots \ldots \quad (353)$$

If we denote the longer arm of the lever G, by L, the shorter arm by l, and the force applied at the extremity of the longer arm by K, we have, from the principle of the lever,

$$K : f :: l : L, \quad \therefore f = \frac{KL}{l} \quad \ldots \ldots \quad (354)$$

Substituting above, we have,

$$F = \frac{PKL}{pl} \quad \ldots \ldots \quad (355)$$

To illustrate, let the area of the larger piston be 100 square inches, that of the smaller piston 1 square inch, the longer arm of the lever

30 inches, the shorter arm 2 inches, and let a force of 100 pounds be applied at the end of the longer arm of the lever; to find the pressure on F.

From the conditions,

$$P = 100,\ K = 100,\ L = 30,\ p = 1, \text{ and } l = 2.$$

Hence,

$$F = \frac{100 \times 100 \times 30}{2} = 150{,}000\ lbs.$$

We have not taken into account the hurtful resistances, hence the pressure of 150,000 pounds must be somewhat diminished.

The volume of water forced from the smaller to the larger cylinder, during a single descent of the piston, B, will occupy, in the two cylinders, spaces whose heights are inversely as the areas of the pistons. Hence, the path, over which f is exerted, is to the path over which F is exerted, as P is to p. Or, denoting these paths by s and S, we have,

$$s : S :: P : p;$$

or, since $P : p :: F : f$, we shall have,

$$s : S :: F : f, \qquad \therefore \quad fs = FS.$$

That is, *the work of the power and resistance are equal,* a principle that holds good in all machines.

EXAMPLES.

1. The cross-section of a sucking and forcing pump is 6 square feet, the play of the piston 3 feet, and the height of the spout, above the reservoir, 50 feet. What must be the effective horse-power of an engine to impart 30 double strokes per minute, hurtful resistances being neglected?

SOLUTION.—The number of units of work required to be performed each minute, is equal to

$$6 \times 3 \times 50 \times 62\tfrac{1}{2} \times 30 = 1{,}687{,}500\ lb.ft.$$

Hence,

$$n = \tfrac{1{,}687{,}500}{33{,}000} = 51\tfrac{4}{33}. \quad Ans.$$

2. In a hydrostatic press, the areas of the pistons are, 2 and 400 square inches, and the arms of the lever are, 1 and 20 inches. Required the pressure on the larger piston for each pound of pressure on the longer arm of lever. *Ans.* 4,000 *lbs.*

3. The areas of the pistons of a hydrostatic press are, 3 and 300 square inches, and the shorter arm of the lever is one inch. What must be the length of the longer arm, that a force of 1 *lb.* may produce a pressure of 1000 *lbs.*? *Ans.* 10 inches.

Storing Up the Work of Hydraulic Pressure.

221. In certain branches of industry it becomes necessary from time to time to handle objects of enormous weight. Work of this kind is most advantageously performed by means of the **hydraulic crane.** The description of the crane itself does not fall within the scope of this work, but the manner in which the motive force is generated and stored up is an immediate application of the principle of the hydraulic press just described. The diagram shows the essential parts of what is called the **accumulator.**

The barrel, *A,* rests firmly on a solid bed-plate, *B,* which also supports the frame *FCE.* A heavy **ram,** *R,* is free to move up and down in the barrel, being guided by a cross-piece, *K,* which is notched on to the posts *E* and *F.* This cross-piece carries a heavy weight, *W,* usually of scrap iron, which may be placed *upon K,* or suspended *from it* in an annular basket. At *P* is a pipe leading *into A,* from a powerful forcing pump, and provided with a valve, *V,* opening inward. At *D* is a pipe leading *from A* to the machine that operates the crane; it may be opened and closed by a stop-cock, *S.*

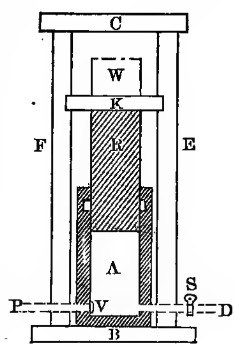

Fig 172.

The stop-cock, *S,* being closed, the pump forces water into

A, raising up the ram and its superposed weight. In some cases the pressure on the water in A is carried as high as 700 *lbs.* on a square inch. When the stored up work is to be applied, the cock S is opened, and the water enters the cylinder, where its enormous pressure is utilized in working the crane.

The principles here explained may be applied to any machine where great force is required for a comparatively short period of time.

The Siphon.

222. The **siphon** is a bent tube, for transferring a liquid from a higher to a lower level, over an intermediate elevation. The siphon consists of two branches, AB and BC, of which the outer one is the longer. To use the instrument, the tube is filled with the liquid, the end of the longer branch being stopped with the finger, or a stop-cock; in which case, the pressure of the atmosphere prevents the liquid from escaping at the other end. The instrument is then inverted, the end, C, being submerged in the liquid, and the stop removed from A. The liquid will flow through the tube, and the flow will continue till the level of the liquid in the reservoir reaches the mouth of the tube, C.

Fig. 173.

To find the velocity with which water will issue from the siphon, let us consider an infinitely small layer at the orifice, A. This layer is pressed downward, by the tension of the atmosphere exerted on the surface of the reservoir, *minus* the weight of the water in the branch, BD, *plus* the weight of the water in the branch, BA. It is pressed upward by the tension of the atmosphere. The difference of these forces, is the weight of the water in DA, and the velocity of the stratum will be due to that height. Denoting the vertical height of DA, by h, we shall have, for the velocity,

$$v = \sqrt{2gh}.$$

This is the theoretical velocity, but it is never quite realized in practice, on account of resistances, that have been neglected in the preceding investigation.

The siphon may be filled by applying the mouth to the end, *A*, and exhausting the air by suction. The tension of the atmosphere, on the upper surface of the reservoir, presses the water up the tube, and fills it, after which the flow goes on as before. Sometimes, a **sucking-tube**, *AD*, is inserted near the opening, *A*, rising to the bend of the siphon. In this case, the opening, *A*, is closed, and the air exhausted through the sucking-tube, *AD*, after which the flow goes on as before.

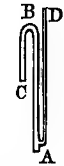

Fig. 174.

The Wurtemburg Siphon.

223. In the **Wurtemburg siphon**, the ends of the tube are bent twice, at right angles, as shown in the figure. The advantage of this is, that the tube, once filled, remains so, as long as the plane of its axis is kept vertical. The siphon may be lifted out and replaced at pleasure, thereby stopping and reproducing the flow at will.

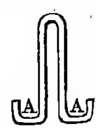

Fig. 175.

It is to be observed that the siphon is only effective when the distance from the highest point of the tube to the level of the water in the reservoir is less than the height at which the atmospheric pressure sustains a column of water in a vacuum. This will, generally, be less than 34 feet.

The Intermitting Siphon.

224. The **intermitting siphon** is represented in the figure. *AB* is a curved tube issuing from the bottom of a reservoir. The reservoir is supplied with water by a tube, *E*, having a smaller bore than the siphon.

To explain its action, suppose the reservoir to be empty, and the tube, E, to be open; as soon as the reservoir is filled to the level, CD, the water begins to flow from the opening, B, and the flow once commenced, continues till the level of the reservoir is reduced to $C'D'$, through the opening, A. The flow then ceases till the cistern is again filled to CD, and so on as before.

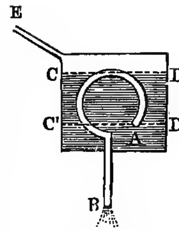

Fig. 176.

The instrument just described is often called **Tantalus' Cup.**

Intermitting Springs.

225. Let A represent a subterranean cavity, communicating with the surface of the earth by a channel, ABC, bent like a siphon. Suppose the reservoir to be fed by percolation through the crevices, or by a small channel, D. When the water in the reservoir rises to the horizontal plane, BD, the flow commences at C, and, if the channel is sufficiently large, the flow continues till the water is reduced to the level plane through C. An intermission then occurs till the reservoir is again filled; and so on, intermittingly.

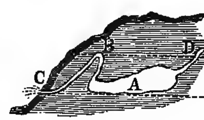

Fig. 177.

Siphon of Constant Flow.

226. We have seen that the velocity of efflux depends on the height of water in the reservoir above the external opening of the siphon. When the water is drawn from the reservoir, the surface sinks, this height diminishes, and, consequently, the velocity continually diminishes.

If, however, the shorter branch, CD, be passed through a cork large enough to float the siphon, the instrument will sink

as the upper surface is depressed, the height of DA will remain constant, and, consequently, the flow will be uniform till the siphon comes in contact with the upper edge of the reservoir. By suitably adjusting the siphon in the cork, the velocity of efflux can be increased or decreased within certain limits. In this manner, any desired quantity of the fluid can be drawn off in a given time.

The siphon is used in the arts, for decanting liquids. It is also employed to draw a portion of a liquid from the interior of a vessel when that liquid is overlaid by one of less specific gravity.

The Hydraulic Ram.

227. The **hydraulic ram** is a machine for raising water by means of shocks caused by the sudden stoppage of a stream of water.

It consists of a reservoir, B, supplied by an inclined pipe, A; at the upper surface of the reservoir, is an orifice closed by a valve, D; this valve is kept in place by a metallic basket immediately below the orifice; G is an air-vessel communicating with the reservoir by an opening F, with a spherical valve, E; this valve closes the orifice F, except when forced upward, in which case its motion is restrained by a framework or cage; H is a delivery-pipe entering the air-vessel at its upper part, and terminating near the bottom.

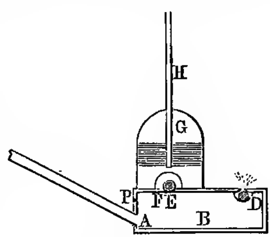

Fig. 178.

To explain the action of the instrument, suppose it empty, and the parts in equilibrium. If a current of water be admitted to the reservoir, through the pipe, A, the reservoir is soon filled, and the water commences rushing out at D; the impulse of the water forces the valve, D, upward, and closes

the opening; the velocity of the water in the reservoir is checked; the reaction forces open the valve, *E*, and a portion of the water enters the air-chamber, *G*; the force of the shock having been expended, the valves both fall by their own weight; a second shock takes place, as before; an additional quantity of water is forced into the air-vessel, and so on continuously. As the water is forced into the air-vessel, the air becomes compressed; and acting by its elastic force, urges a stream of water up the pipe, *H*. The shocks occur in rapid succession, and thus a constant stream is kept up.

To explain the use of the valve, *P*, it may be remarked that water absorbs more air under a greater, than under a smaller pressure. Hence, as it passes through the air-chamber, a portion of the contained air is taken up by the water and carried out through the pipe, *H*. But each time that the valve, *D*, falls, there is a tendency to a vacuum in the upper part of the reservoir, in consequence of the rush of the fluid to escape through the opening. The pressure of the external air then forces open the valve *P*, a portion of air enters, and is afterward forced up with the water into the vessel, *G*, to keep up the supply.

The hydraulic ram is only used to raise small quantities of water, as for the supply of a house, or garden. Only a small fraction of the fluid that enters the supply-pipe actually passes out through the delivery-pipe; but if the head of water is pretty large, a column may be raised to a great height. Water is often raised, in this manner, to the highest parts of lofty buildings.

Sometimes, an additional air-vessel is introduced over the valve, *E*, to deaden the shock of the valve in its play.

Archimedes' Screw.

228. This is a machine for raising water through small heights, and, in its simplest form, it consists of a tube wound

spirally around a cylinder. The cylinder is mounted so that its axis is oblique to the horizon, the lower end dipping into the reservoir. When the cylinder is turned on its axis, the lower end of the tube describes the circumference of a circle, whose plane is perpendicular to the axis. When the mouth of the tube comes to the level of the axis and begins to ascend, there is a certain quantity of water in the tube, which continues to occupy the lowest part of the spire; and, if the cylinder is properly inclined to the horizon, this flow is toward the upper end of the tube. At each revolution, a quantity of water enters the tube, and that already in the tube is raised, higher and higher, till, at last, it flows from the upper end of the tube.

The Chain Pump.

229. The **chain pump** is an instrument for raising water through small elevations.

It consists of an endless chain passing over wheels, *A* and *B*, having their axes horizontal, one below the surface of the water, and the other above the spout of the pump. Attached to this chain, and at right angles to it, are circular disks, fitting the tube, *CD*. If the cylinder *A*, be turned in the proper direction, the buckets or disks rise through the tube *CD*, driving the water before them, until it reaches the spout *C*, and escapes. One great objection to this machine is, the difficulty of making the disks fit the tube. Hence, there is a constant leakage, requiring great additional expenditure of force.

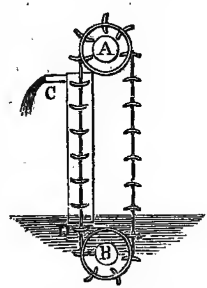

Fig. 179.

Sometimes the body of the pump is inclined, in which case it does not differ much in principle from a wheel with flat buckets, which has also been used for raising water.

Blowers.

230. A **blower** is a machine for creating and keeping up a constant flow of air, either for promoting combustion, or for the purposes of ventilation. One of the simplest, and not the least important, of this class of machines, is the blacksmith's bellows, a section of which is shown in Fig. 180.

The Blacksmith's Bellows.

231. The **blacksmith's bellows** is a machine that resembles the accumulator described in Art. 221, air being used instead of water. It consists essentially of three plates, usually of wood, A, B, and C. The middle one, A, is fixed in position, whilst the upper and lower ones, C and B, are free to turn about hinge-joints at E and D; the upper and lower plates are attached to the middle one by flexible sheets of leather, nailed to their respective edges; the joints at E and D are also covered by pieces of leather attached to the plates in the same manner. By this arrangement the whole machine is separated into two air-tight compartments, U and L, the capacity of each being capable of a certain amount of contraction and expansion.

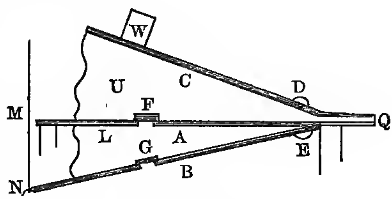

Fig. 180.

The middle plate, A, has a valve, F, opening upward and closing by its own weight; a similar valve, G, exists in the lower plate. The upper compartment terminates in a **nozzle** through which the air contained in the compartment U, is forced by the action of the weight, W. The supply of air in the upper compartment is kept up by means of the movable plate B, which is raised by means of a link, MN, worked by

a lever not shown in the diagram, and which falls by its own weight. When the plate *B*, is raised all the air in the compartment *L*, is forced into the compartment *U*, where it is kept from returning by the closing of the valve *F*; whilst the plate *B*, is descending the valve *G*, is kept open by the pressure of the external air and the compartment *L*, is filled. When the plate *B*, is raised again a new supply of air is forced into the compartment *U*, and so on indefinitely.

The strength of the current through the nozzle may be increased by increasing the weight, *W*.

The Ventilating Blower.

232. One form of **ventilating blower** is shown in Fig. 181. A solid piston, *P*, is moved up and down in the barrel, *A*, by a piston-rod, *R*, set in motion by a steam-engine. The barrel, *A*, is provided with four valves, *B*, *C*, *D*, *E*, the first two of which open inward and the others open outward, as indicated in the diagram. Air passing through either *D* or *E* is driven through the pipe, *F*, to the place where it is to be utilized.

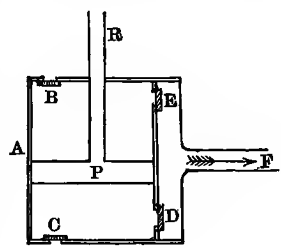

Fig. 181.

To explain the action of this machine, let us suppose the piston to be, say, at the upper limit of its play. Then, when the piston is depressed, the valve, *C*, will be closed by the tension of the compressed air, the valve, *D*, will be opened, the air which is forced into the pipe will close the valve, *E*, and all the air below the piston will be forced into the pipe, *F*; meantime the valve, *B*, is opened by the pressure of the external air and the space behind the piston is filled with air from without. When the piston is raised the valve, *B*, is closed by the tension of the compressed air, the valve, *E*, is

opened, the air which is forced into the pipe closes the valve, *D*, and all the air above the piston is forced into the pipe, *F*; meantime the valve, *C*, is opened by the pressure of the external air and the space behind the piston is filled with air from without. In this manner a continuous stream of air is kept flowing through the pipe, *F*, which may be utilized in any desirable manner.

The Air Pump.

233. The **air pump** is a machine for rarefying air.

It consists of a barrel, *A*, in which a piston, *B*, is worked up and down by a lever, *C*, attached to a piston-rod, *D*. The barrel communicates with a vessel, *E*, called a **receiver**, by a narrow pipe. The receiver is usually of glass, ground to fit air-tight on a smooth bed-plate, *KK*. The joint between the receiver and plate may be rendered more perfectly air-tight by interposing a layer of tallow. A stop-cock, *H*, permits communication to be made at pleasure between the barrel and receiver, or between the barrel and external air. When the stop-cock is turned in a particular direction, the barrel and receiver communicate; but on turning it through 90 degrees, the communication with the receiver is cut off, and a communication is opened between the barrel and external air. Instead of the stop-cock, valves are often used, that are opened and closed by the elastic force of the air, or by the force that works the pump. The communicating pipe should be exceedingly small, and the piston, *B*, when at its lowest point, should fit accurately into the bottom of the barrel.

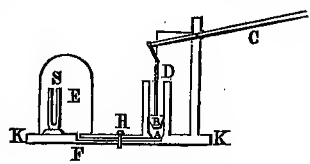

Fig. 182.

To explain the action of the air pump, suppose the piston to be at its lowest position. The stop-cock, *H*, is turned so

as to open a communication between the barrel and receiver, and the piston is raised to its highest point by a force applied to the lever, C. The air, which before occupied the receiver and pipe, expands so as to fill the barrel, receiver, and pipe. The stop-cock is then turned to cut off communication between the barrel and receiver, and open the barrel to the external air, and the piston again depressed to its lowest position. The air in the barrel is expelled by the depression of the piston. The air in the receiver is now more rare than at the beginning, and by a continued repetition of the process, any degree of rarefaction may be attained.

To measure the rarefaction of the air in the receiver, a siphon-gauge may be used, or a glass tube, 30 inches long, may be made to communicate at its upper extremity with the receiver, whilst its lower extremity dips into a cistern of mercury. As the air is rarefied in the receiver, the pressure on the mercury in the tube becomes less than on that in the cistern, and the mercury rises in the tube. The tension of the air in the receiver is indicated by the difference between the height of the barometric column and that of the mercury in the tube.

To investigate a formula for the tension of the air in the receiver, after any number of double strokes, let us denote the capacity of the receiver by r, that of the connecting-pipe by p, and that of the space between the bottom of the barrel and the highest position of the piston by b. Denote the original tension of the air by t; its tension after the first upward stroke of the piston by t'; after the second, third, ... n^{th}, upward strokes, by t'', t''', ... $t^{n'}$.

The air which occupied the receiver and pipe, after the first upward stroke, fills the receiver, pipe, and barrel: according to MARIOTTE's law, its tension in the two cases varies inversely as the volumes occupied; hence,

$$t : t' :: p + r + b : p + r, \quad \therefore\ t' = t\frac{p + r}{p + r + b}.$$

In like manner, we shall have, after the second upward stroke,

$$t' : t'' :: p + r + b : p + r, \quad \therefore\ t'' = t'\frac{p + r}{p + b + r}.$$

Substituting for t' its value, deduced from the preceding equation, we have,

$$t'' = t\left(\frac{p + r}{p + b + r}\right)^2.$$

In like manner, we find,

$$t''' = t\left(\frac{p + r}{p + b + r}\right)^3;$$

and, in general, after the n^{th} stroke,

$$t^{n\prime} = t\left(\frac{p + r}{p + b + r}\right)^n.$$

If the pipe is exceedingly small, its capacity may be neglected in comparison with that of the receiver, and we then have,

$$t^{n\prime} = t\left(\frac{r}{b + r}\right)^n.$$

Let it be required, for example, to determine the tension of the air after 5 upward strokes, when the capacity of the barrel is one third that of the receiver.

In this case, $\dfrac{r}{b + r} = \frac{3}{4}$, and $n = 5$, whence,

$$t^{v} = t\,\tfrac{243}{1024}.$$

Hence, the tension is less than a fourth part of that of the external air.

Instead of the receiver, the pipe may be connected by a screw-joint with any closed vessel, as a hollow globe, or glass flask. In this case, by reversing the direction of the stop-cock, in the up and down motion of the piston, the instrument may be used as a condenser. When so used, the tension after n downward strokes of the piston, is given by the formula,

$$t^{n'} = t\left(\frac{r + nb}{r}\right).$$

Taking the same case as that before considered, with the exception that the instrument is used as a condenser instead of a rarefier, we have, after 5 downward strokes,

$$t^{v} = \tfrac{8}{3}t.$$

That is, the tension is eight thirds that of the external air.

Artificial Fountains.

234. An **artificial fountain** is an instrument by which a liquid is forced upward in the form of a jet, by the tension of condensed air. The simplest form of artificial fountain is called HERO'S ball.

Hero's Ball.

235. This consists of a globe, A, into the top of which is inserted a tube, B, reaching nearly to the bottom of the globe. This tube is provided with a stop-cock, C, by which it may be closed, or opened at pleasure. A second tube, D, enters the globe near the top, which is also provided with a stop-cock, E.

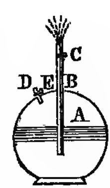

Fig. 183.

To use the instrument, close the stop-cock, C, and fill the lower portion of the globe with water through D; then connect D with a condenser, and pump air

into the upper part of the globe, and confine it there by closing the stop-cock, *E*. If, now, the stop-cock, *C*, be opened, the pressure of the confined air on the surface of the water in the globe forces a jet through the tube, *B*. This jet rises to a greater or less height, according to the greater or less quantity of air that was forced into the globe. The water will continue to flow through the tube as long as the tension of the confined air is greater than that of the external atmosphere, or else till the level of the water in the globe reaches the lower end of the tube.

Instead of using the condenser, air may be introduced by blowing with the mouth through the tube, *D*, and confined by turning the stop-cock, *E*.

The principle of HERO'S ball is the same as that of the air-chamber in the forcing-pump and fire-engine, already explained.

Hero's Fountain.

236. Hero's fountain is constructed on the same principle as HERO'S ball, except that the compression of the air is effected by the weight of a column of water, instead of by a condenser.

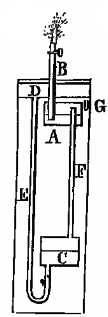

Fig. 184.

A is a cistern, similar to HERO'S ball, with a tube, *B*, extending nearly to the bottom of the cistern. *C* is a second cistern placed at some distance below *A*. This cistern is connected with a basin, *D*, by a bent tube, *E*, and also with the upper part of the cistern, *A*, by a tube, *F*. When the fountain is to be used, *A* is nearly filled with water, *C* being empty. A quantity of water is then poured into the basin, *D*, which, acting by its weight, sinks into *C*, compressing the air in the upper portion of it into a smaller space, thus increasing its tension. This increase of tension acting on the surface of the water in *A*, forces a jet through *B*, which rises to a greater or less

height according to the greater or less tension. The flow will continue till the level of the water in A reaches the bottom of the tube, B. The measure of the compressing force on a unit of surface of the water in C, is the weight of a column of water, whose base is that unit, and whose altitude is the difference of level between the water in D and in C.

If HERO's ball be partially filled with water and placed under the receiver of an air-pump, the water will be observed to rise in the tube, forming a fountain, as the air in the receiver is exhausted. The principle is the same as before; the flow is due to an excess of pressure on the water within the globe over that without. In both cases, the flow is resisted by the tension of the air without, and is urged on by the tension within.

Wine-Taster and Dropping-Bottle.

237. The **wine-taster** is used to bring up a small portion of wine or other liquid from a cask. It consists of a tube, open at the top, and terminating below in a narrow tube, also open. When it is to be used, it is inserted to any depth in the liquid, which rises in the tube to the level of the liquid without. The finger is then placed so as to close the upper end of the tube, and the instrument raised out of the cask. The fluid escapes from the lower end, until the pressure of the rarefied air in the tube, *plus* the weight of a column of liquid, whose cross-section is that of the tube, and whose altitude is that of the fluid retained, is equal to the pressure of the external air. If the tube be placed over a tumbler, and the finger removed from the upper orifice, the fluid brought up flows into the tumbler.

Fig. 185.

If the lower orifice is very small, a few drops may be allowed to escape, by taking off the finger and immediately replacing it. The instrument then constitutes the **dropping-bottle.**